

Group Equivariant Deep Learning

Lecture 2 - Steerable group convolutions

Lecture 2.1 - Steerable kernels/basis functions

Definition and $SO(2)$ example

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Definition and $SO(2)$ example

Lecture 2.2 - Revisiting regular G-convs with steerable kernels | Template matching viewpoint

Motivating the Fourier transform on and showing we now no longer need a grid on the sub-group H !

Lecture 2.3 - **Group Theory** | Irreducible representations and Fourier transform

Preliminaries for steerable feature fields and steerable g-conv intuition with a focus on $SO(2)$

Lecture 2.4 - **Group Theory** | Induced representations and feature fields

Preliminaries (and intuition) for steerable group convolutions

Lecture 2.5 - Steerable group convolutions

And how to use them

Lecture 2.6 - Activation functions for steerable G-CNNs

Examples of which we can and cannot use

Lecture 2.7 - Derivation of Harmonic networks¹ from regular g-convs | Recalling g-convs are all you need!

¹Worrall, D. E., Garbin, S. J., Turmukhambetov, D., & Brostow, G. J. Harmonic networks: Deep translation and rotation equivariance. CVPR 2017

Steerable basis

A vector $Y(x) = \begin{pmatrix} \vdots \\ Y_l(x) \\ \vdots \end{pmatrix} \in \mathbb{K}^L$ with (basis) functions $Y_l \in \mathbb{L}_2(X)$ is steerable if

$$\forall_{g \in G} : \quad Y(g x) = \rho(g) Y(x),$$

where $g x$ denotes the action of G on X and $\rho(g) \in \mathbb{K}^{L \times L}$ is a representation of G .

I.e., we can transform all basis functions simply by taking a linear combination of the original basis functions.

Example: Steerable basis on S^1 (circular harmonics)

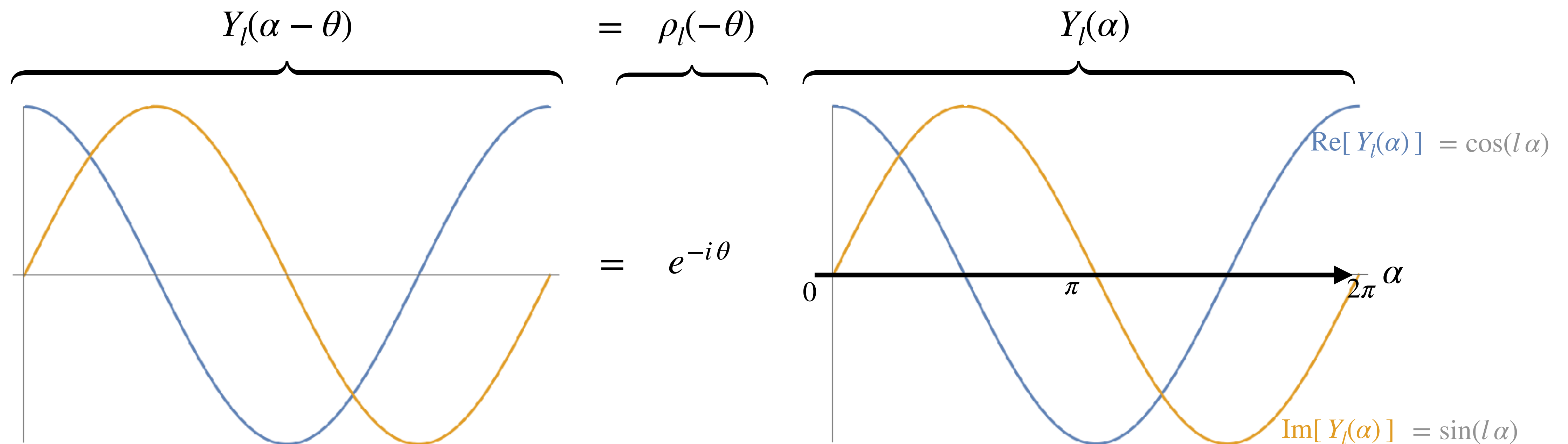
Basis functions (for $\mathbb{L}_2(S^1)$): $Y_l(\alpha) = e^{il\alpha}$
Are steered by representations: $\rho_l(\theta) = e^{il\theta}$

$$\begin{aligned} \text{Proof: } Y_l(\alpha - \theta) &= e^{il(\alpha - \theta)} \\ &= e^{-il\theta} e^{il\alpha} \\ &= \rho_l(-\theta) Y_l(\alpha) \end{aligned}$$

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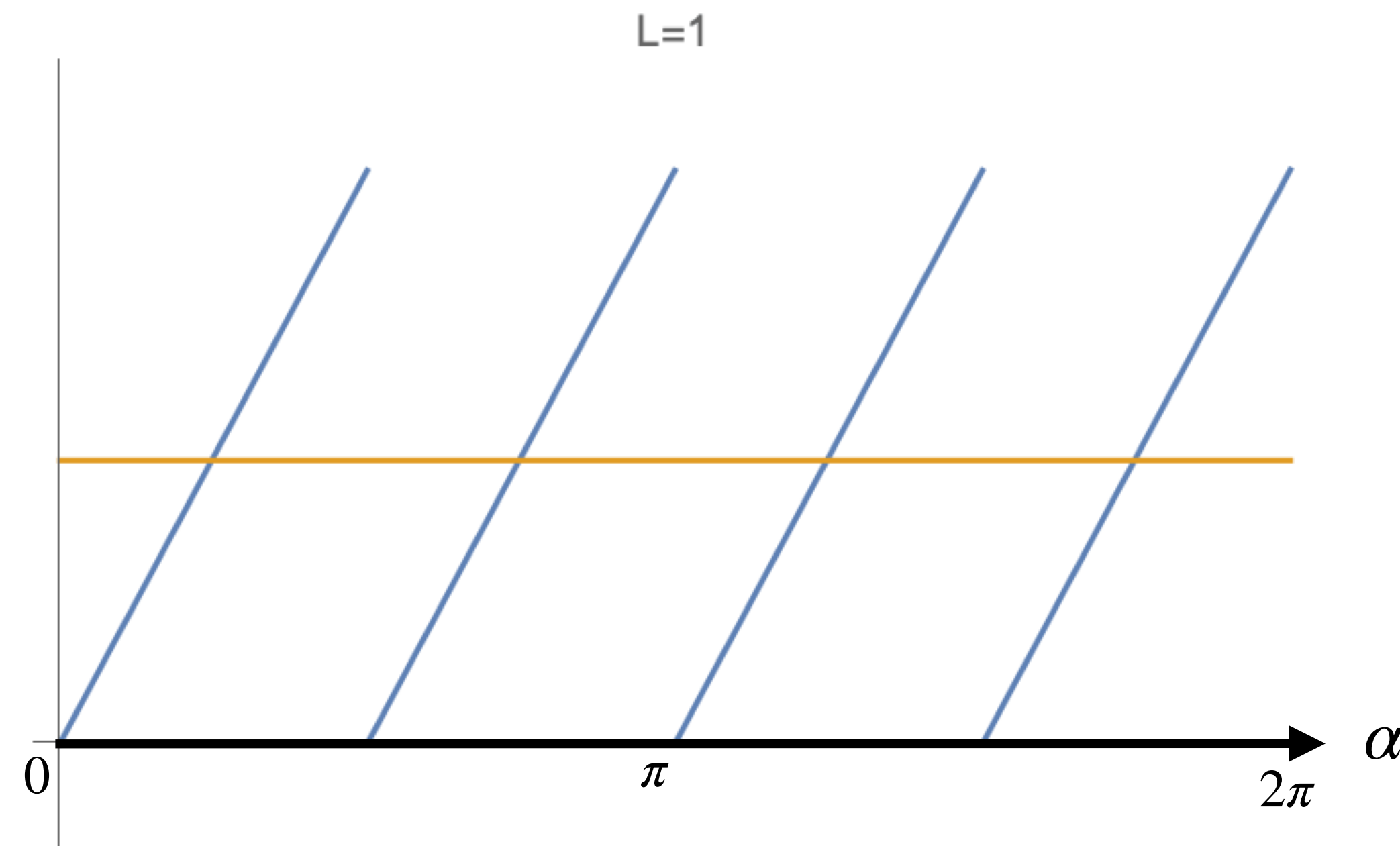
Basis functions (for $\mathbb{L}_2(S^1)$):

Form a complete orthonormal (Fourier) basis:

Y_l are given by the irreps of $SO(2)$ and hence form orthogonal basis (Peter-Weyl Theorem)

$$Y_l(\alpha) = e^{i l \alpha}$$

$$f(\alpha | \hat{\mathbf{w}}) = \sum_{l=-\infty}^{\infty} \overline{\hat{w}_l} Y_l(\alpha)$$



$$f(\alpha) = \alpha \bmod \pi/2$$

$$f(\alpha | \hat{\mathbf{w}}) = \sum_{l=-L}^L \overline{\hat{w}_l} Y_l(\alpha)$$

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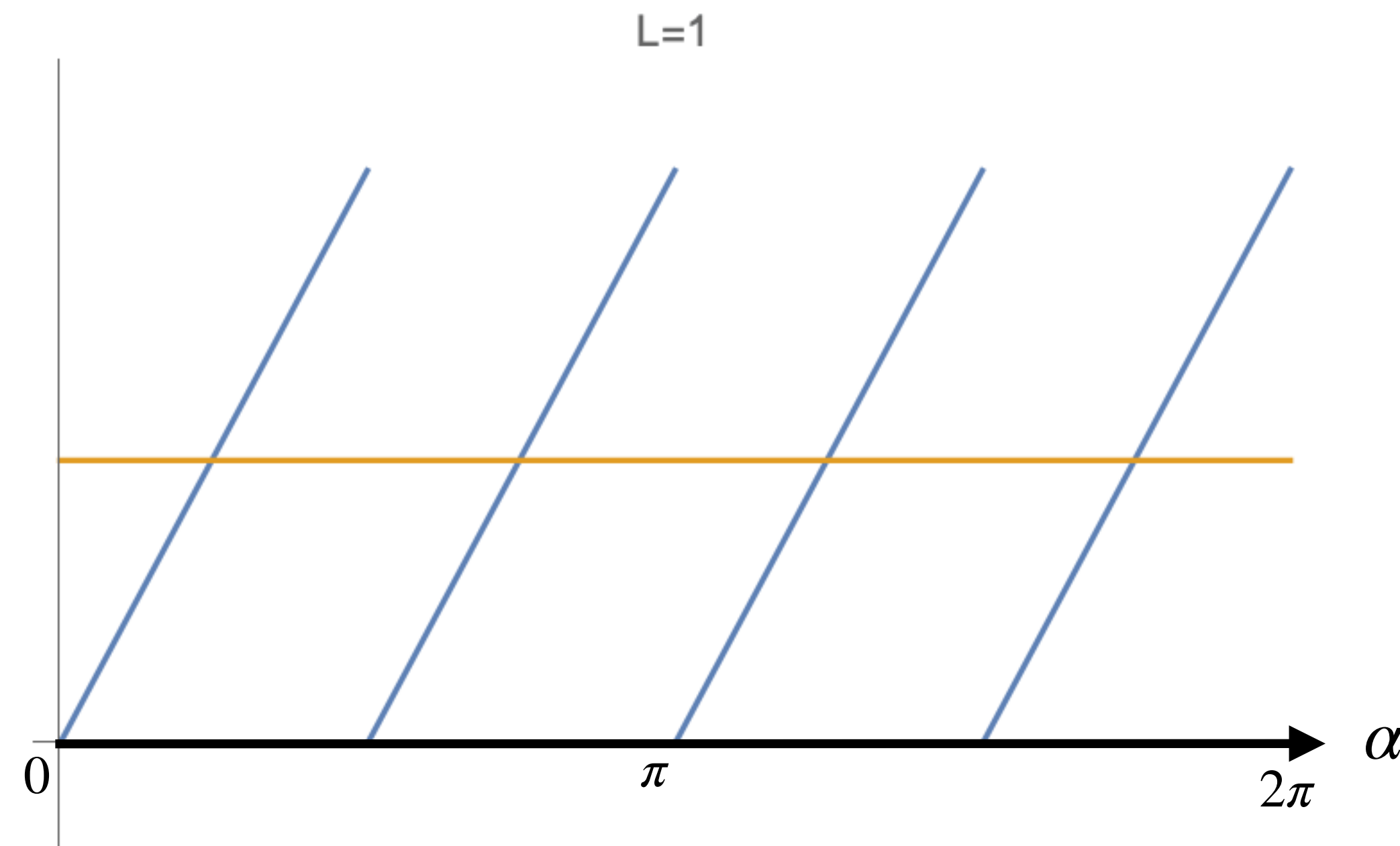
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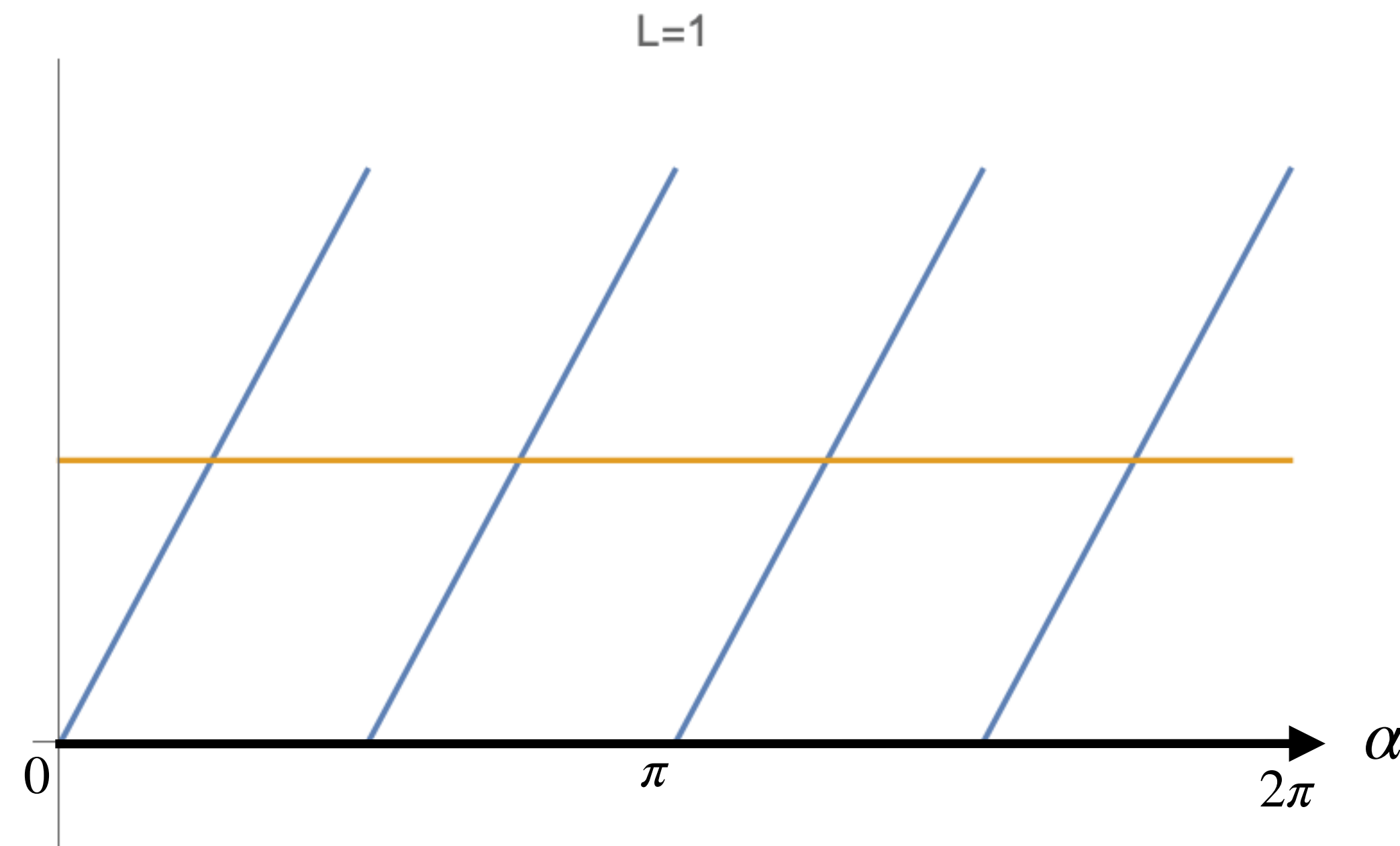
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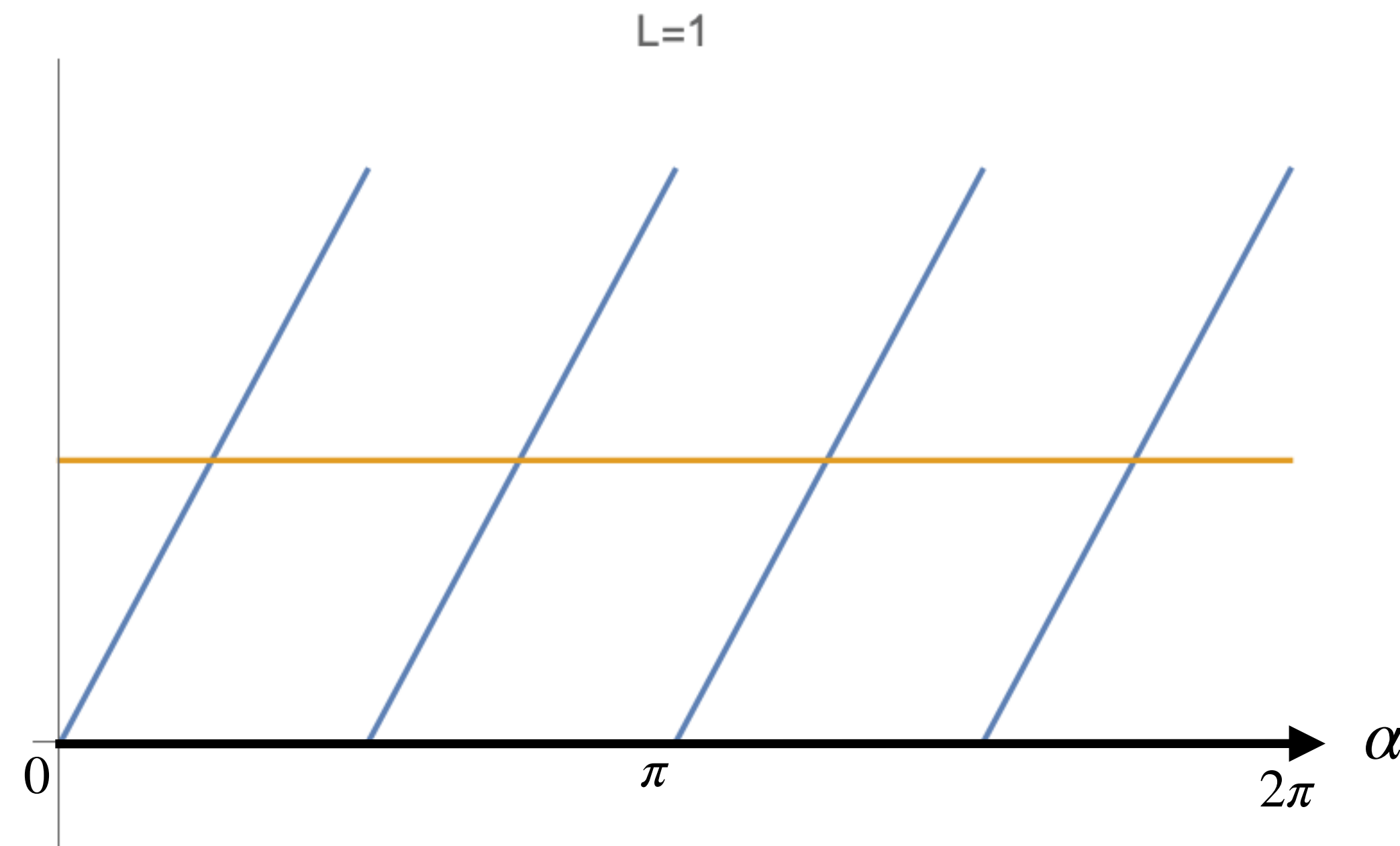
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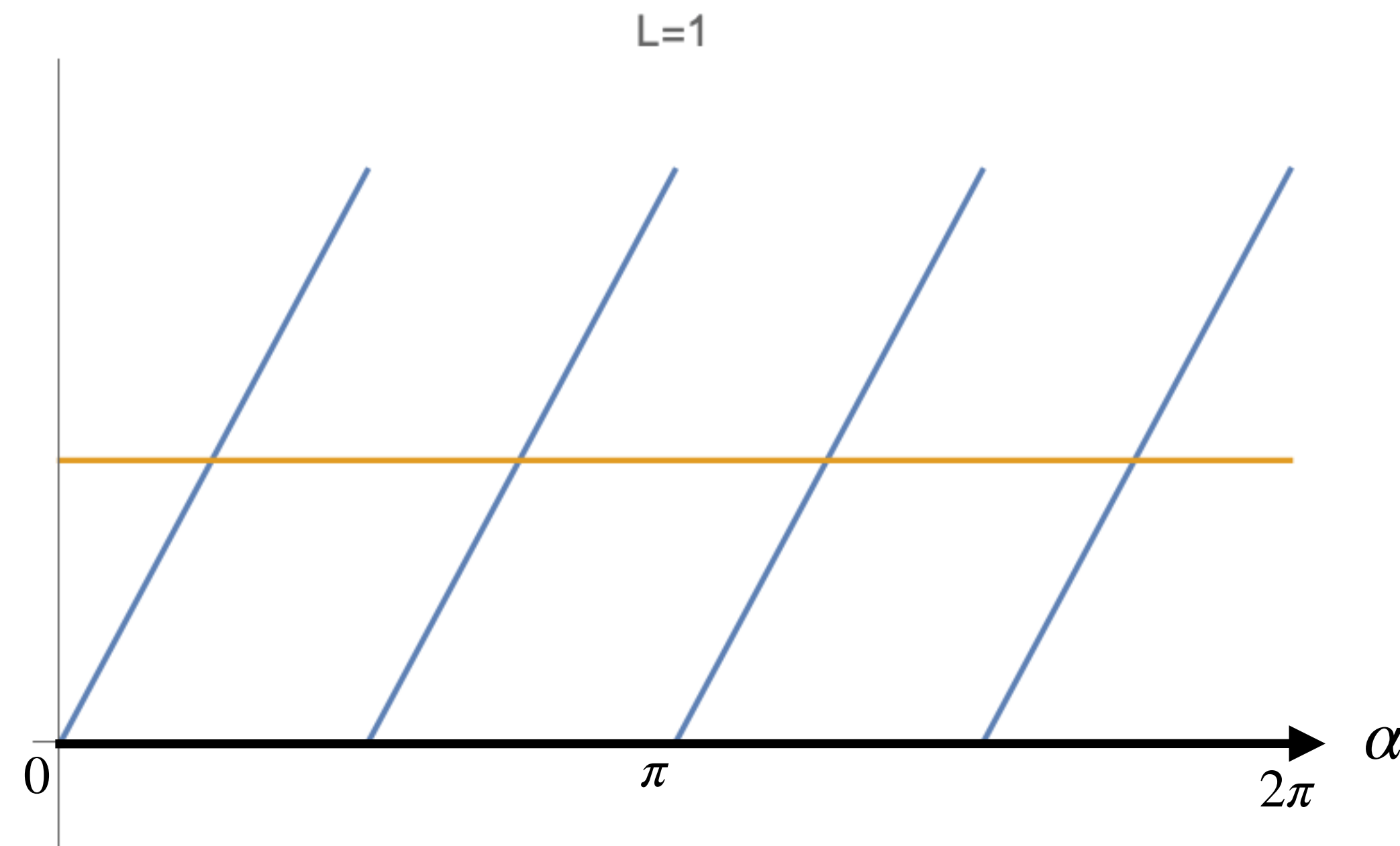
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Example: Steerable basis on S^1 (circular harmonics)

$Y(\alpha - \theta)$

$$\rho(-\theta) = \bigoplus_{l=-L}^L \rho_l(-\theta)$$

$Y(\alpha)$

$$\begin{pmatrix} \text{Plot 1} \\ \text{Plot 2} \\ \text{Plot 3} \\ \text{Plot 4} \\ \text{Plot 5} \\ \text{Plot 6} \\ \text{Plot 7} \end{pmatrix} = \begin{pmatrix} e^{i3\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i2\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i1\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i1\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i2\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-i3\theta} \end{pmatrix} \begin{pmatrix} \text{Plot 1} \\ \text{Plot 2} \\ \text{Plot 3} \\ \text{Plot 4} \\ \text{Plot 5} \\ \text{Plot 6} \\ \text{Plot 7} \end{pmatrix}$$

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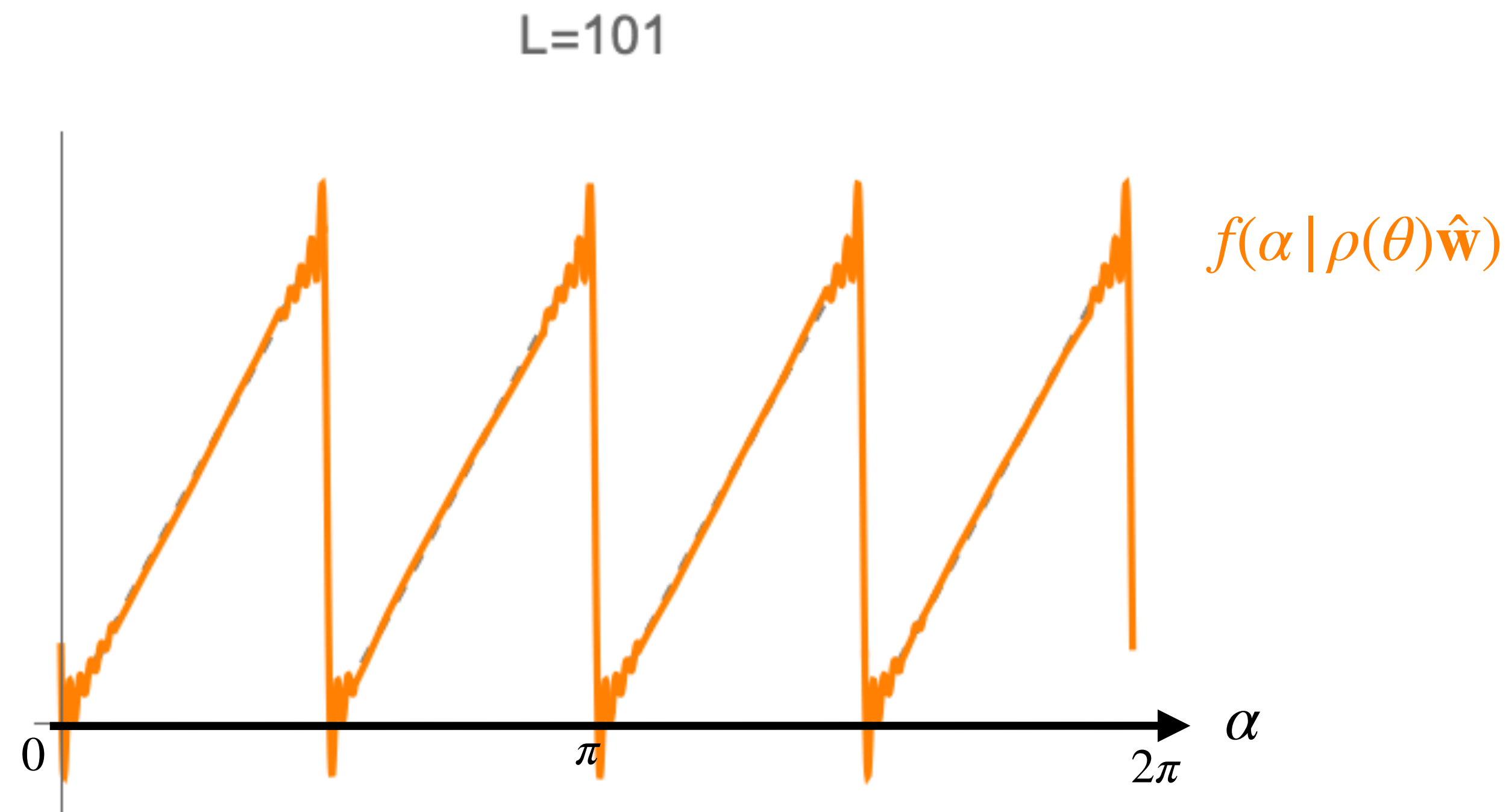
Let $f(\alpha | \hat{\mathbf{w}}) = \hat{\mathbf{w}}^\dagger Y(\alpha)$

Then we can **steer**/shift this function **by transforming the weights $\hat{\mathbf{w}}$**

$$f(\alpha - \theta | \hat{\mathbf{w}}) = f(\alpha | \rho(\theta)\hat{\mathbf{w}})$$

Proof:

$$\begin{aligned} f(\alpha - \theta | \hat{\mathbf{w}}) &= \hat{\mathbf{w}}^\dagger Y(\alpha - \theta) \\ &= \hat{\mathbf{w}}^\dagger \rho(-\theta) Y(\alpha) \\ &= \hat{\mathbf{w}}^\dagger \rho(\theta)^\dagger Y(\alpha) \\ &= (\rho(\theta)\hat{\mathbf{w}})^\dagger Y(\alpha) \\ &= f(\alpha | \rho(\theta)\hat{\mathbf{w}}) \end{aligned}$$



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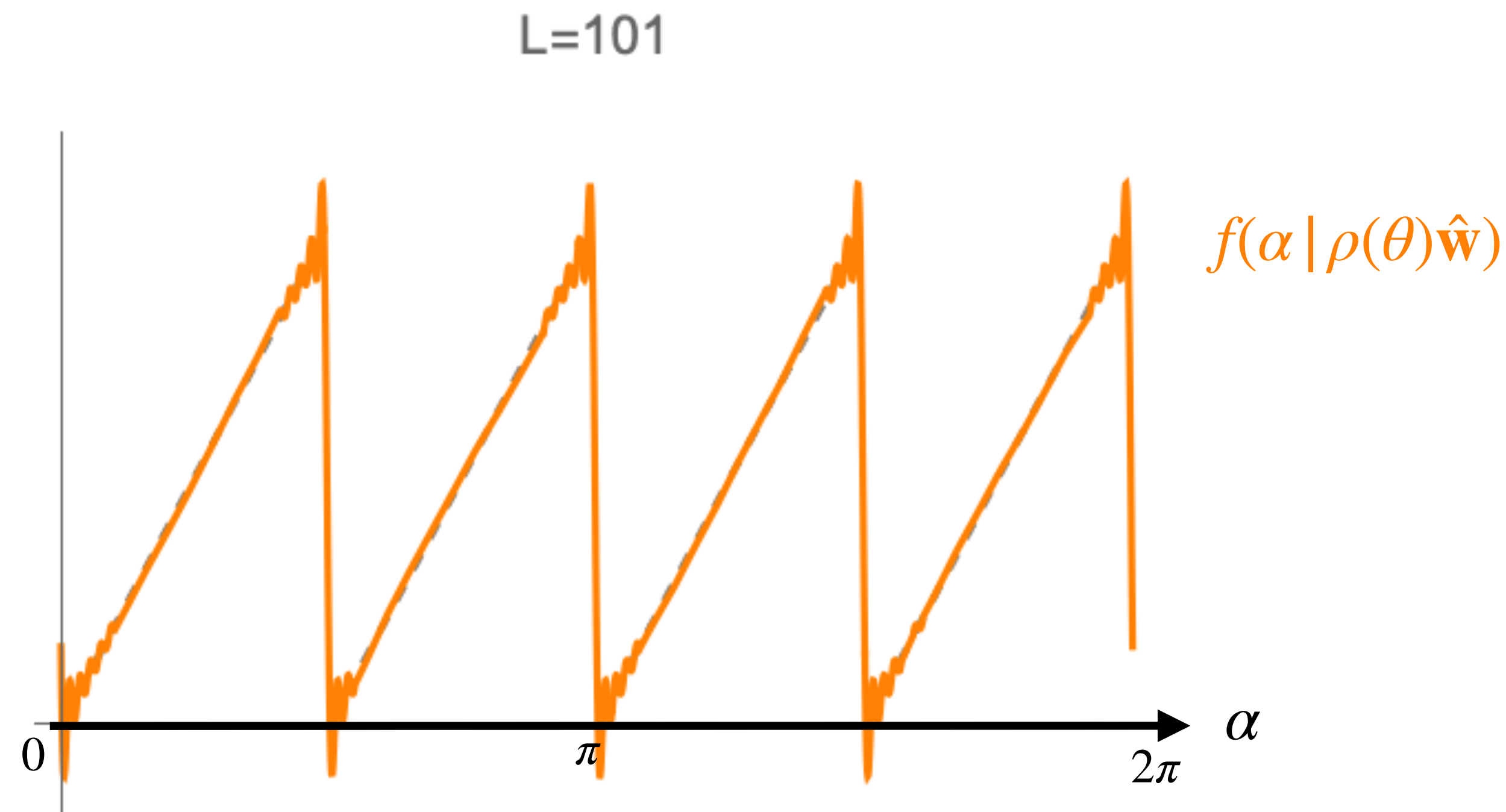
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Two dimensional rotation-steerable functions

- The previous functions $\rho_l(\theta) = e^{il\theta}$ are (irreducible) representations of $SO(2)$

Recall lecture 1.6 (Group Theory | Homogeneous/quotient spaces)

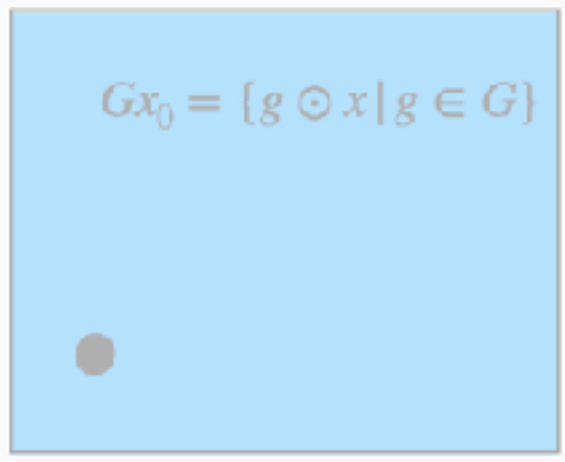
- The group $SO(2)$ can also act on \mathbb{R}^2
 - Though not transitively...
 - It does act transitively on S^1 though

Transitive action

Transitive action: An action $\odot : G \times X \rightarrow X$ such that


$$\forall x_0, x \in X \exists g \in G : x = g \odot x_0$$

$(\mathbb{R}^2, +)$ acts transitively on \mathbb{R}^2

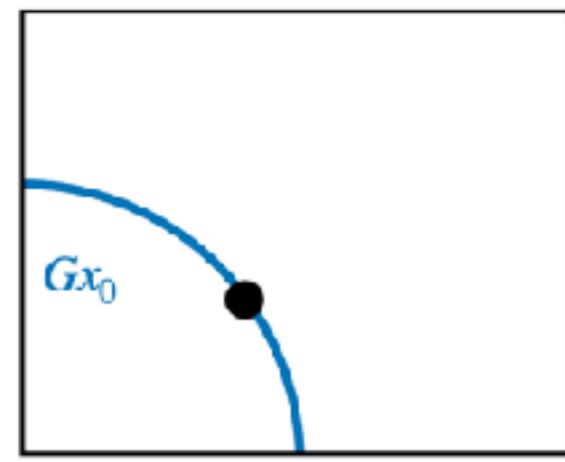


$Gx_0 = \{g \odot x \mid g \in G\}$

$SE(2)$ acts transitively on \mathbb{R}^2



$SO(2)$ does not ...



- Use polar coordinates $\mathbb{R}^2 \ni \mathbf{x} \leftrightarrow (r, \alpha) \in \mathbb{R}^+ \times S^1$ to come up with a rotation-steerable basis for $\mathbb{L}_2(\mathbb{R}^2)$!

Two dimensional rotation-steerable functions

- Consider a function $f(\mathbf{x}) = \bar{f}(r, \alpha)$ in polar coordinates

$$\mathbf{x} = (r \cos \alpha, r \sin \alpha)$$

- The action of $SO(2)$ on \mathbb{R}^2 in polar coords translates to

$$\mathbf{x} \mapsto \mathbf{R}_\theta \mathbf{x} \quad \leftrightarrow \quad (r, \alpha) \mapsto (r, \alpha + \theta)$$

Proof: $\mathbf{R}_\theta \mathbf{x} = \mathbf{R}_\theta \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$
 $= \begin{pmatrix} r(\cos \theta \cos \alpha - \sin \theta \sin \alpha) \\ r(\cos \theta \sin \alpha + \sin \theta \cos \alpha) \end{pmatrix}$
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- Then, functions are rotated simply by a shift in the angular axis

$$\mathcal{L}_\theta^{SO(2)} f(\mathbf{x}) = f(\mathbf{R}_\theta^{-1} \mathbf{x}) \quad \leftrightarrow \quad \mathcal{L}_\theta^{SO(2)} \bar{f}(r, \alpha) = \bar{f}(r, \alpha - \theta)$$

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- Now, let's use this to parametrize polar-separable conv kernels and focus on the angular part

$$k(\mathbf{x} | \mathbf{w}) = k^\rightarrow(r | \mathbf{w}) k^\cup(\alpha | \mathbf{w})$$

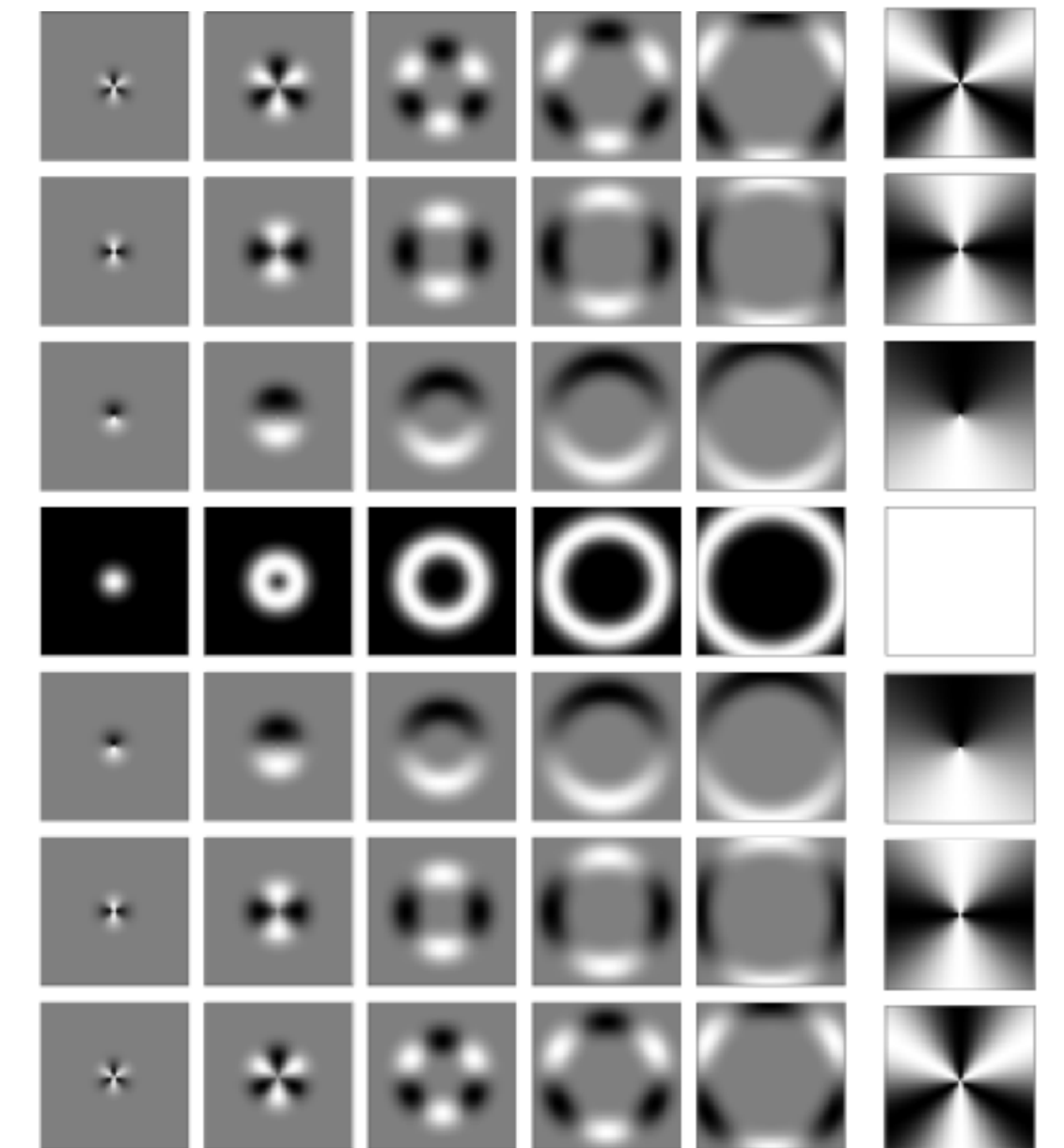
A function on S^1 !!!

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$$k_m^\rightarrow(r)$$



Two dimensional rotation-steerable functions

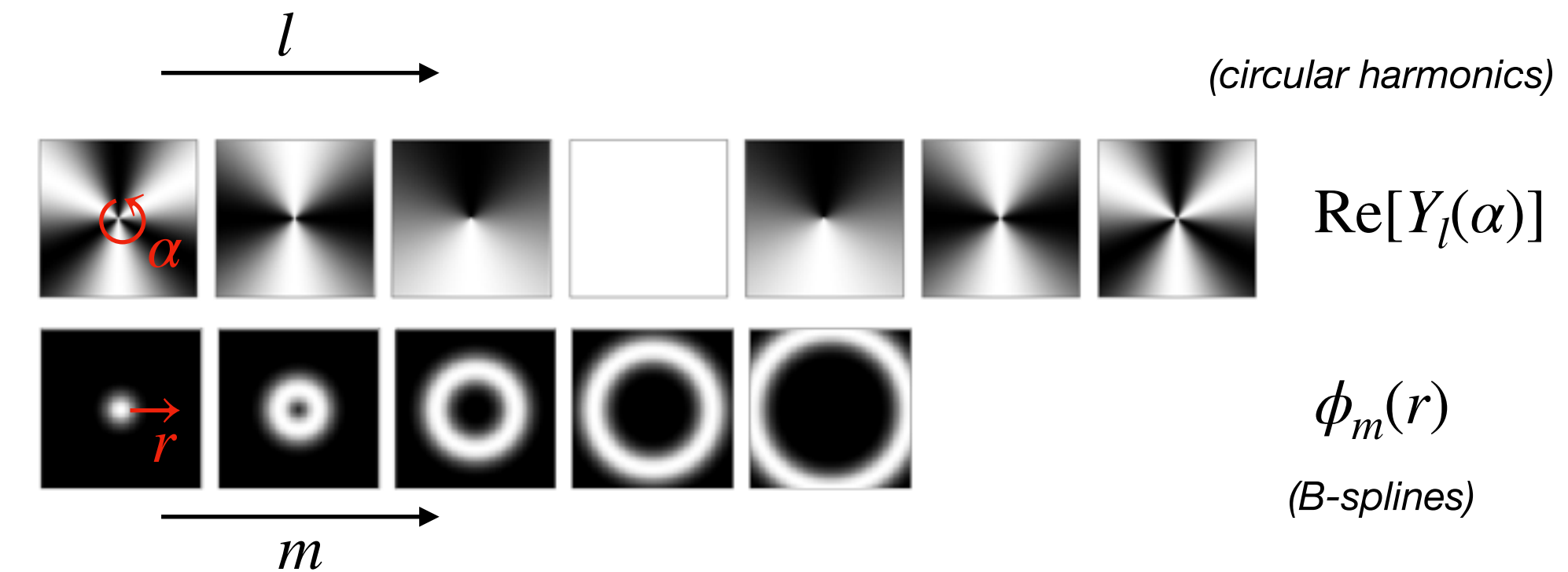
- Consider polar-separable convolution kernel:

$$k(\mathbf{x} | \mathbf{w}) = k^{\rightarrow}(r | \mathbf{w}) k^{\circ}(\alpha | \mathbf{w}),$$

- with k° in an $SO(2)$ steerable basis, and k^{\rightarrow} in some radial basis:

$$k^{\circ}(\alpha | \mathbf{w}) = \sum_l \bar{w}_l Y_l(\alpha), \quad \text{e.g., with} \quad Y_l(\alpha) = e^{il\alpha},$$

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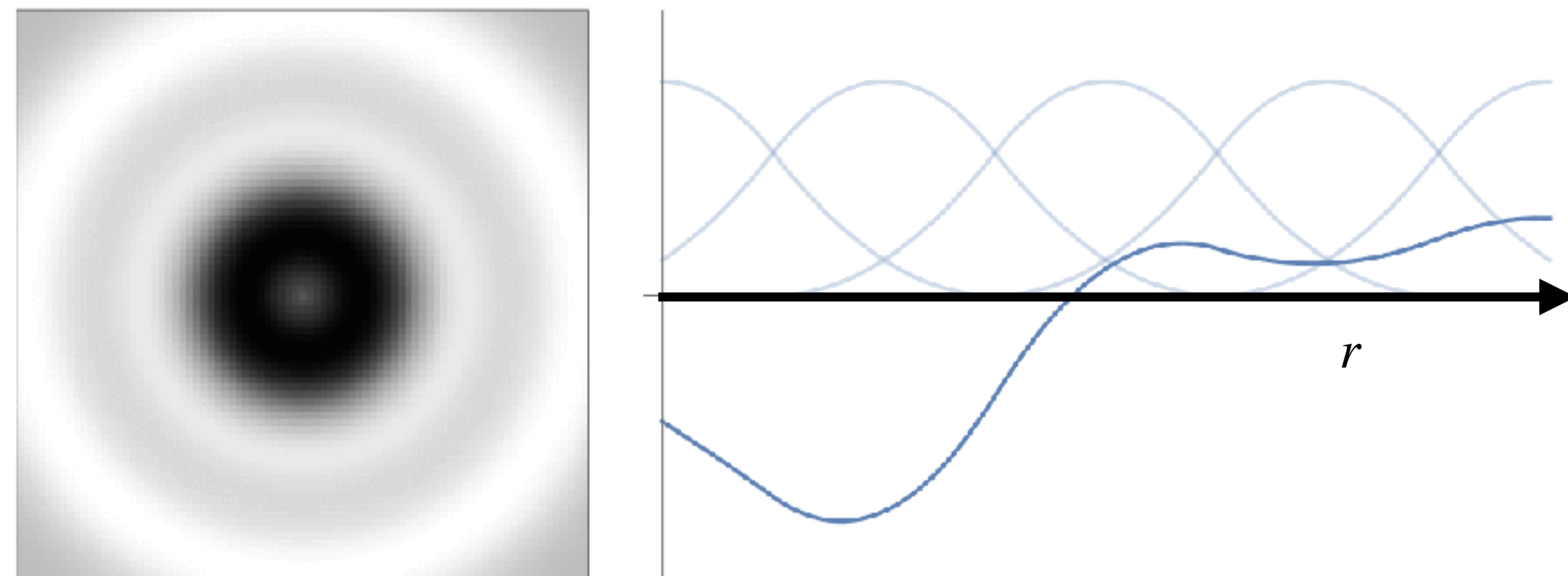
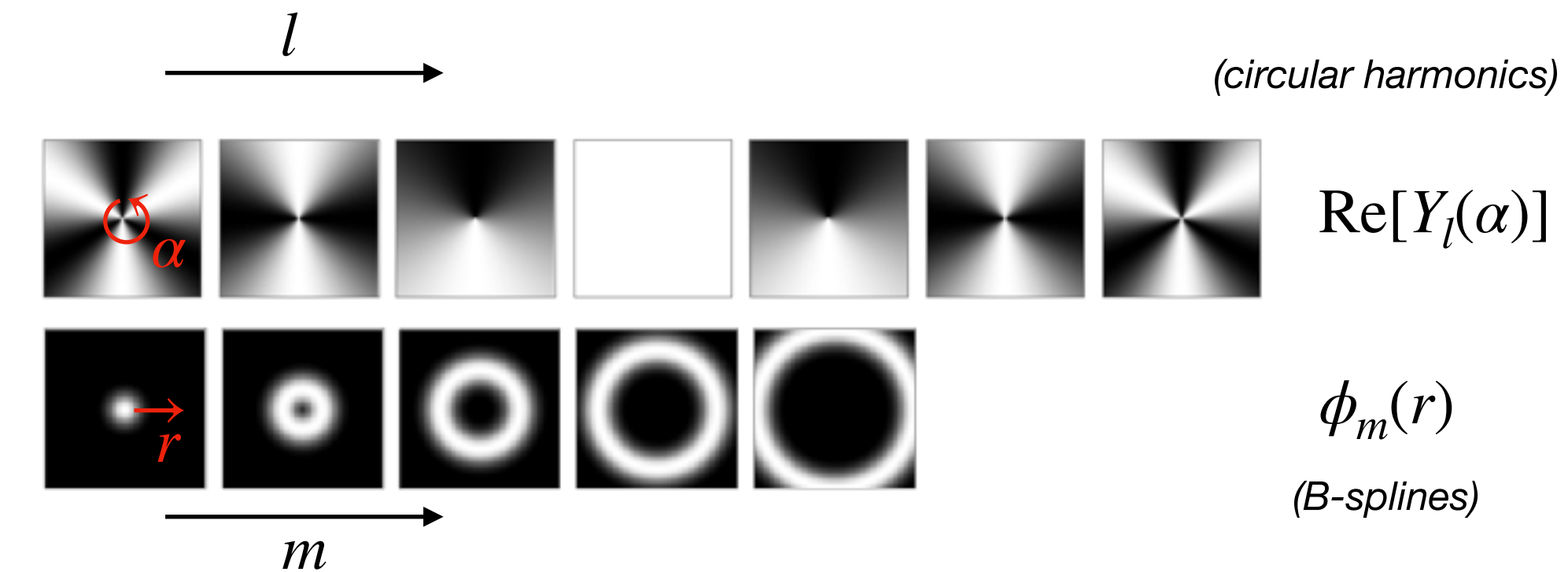
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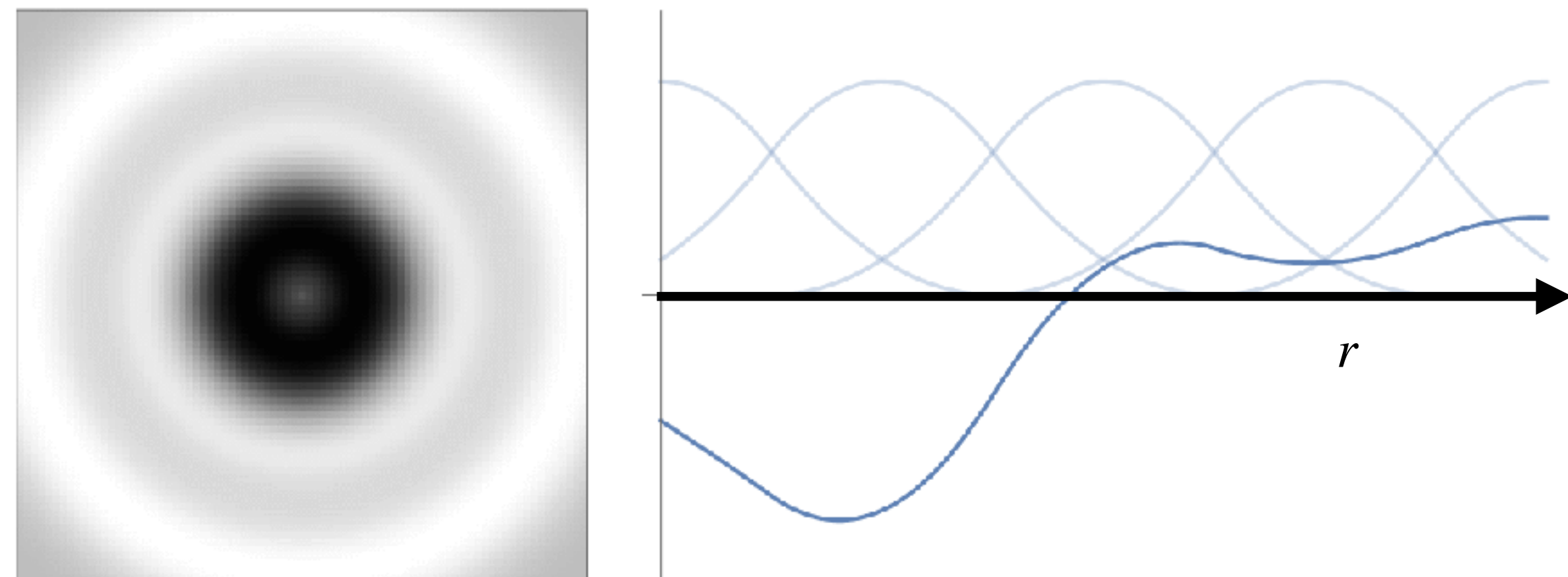
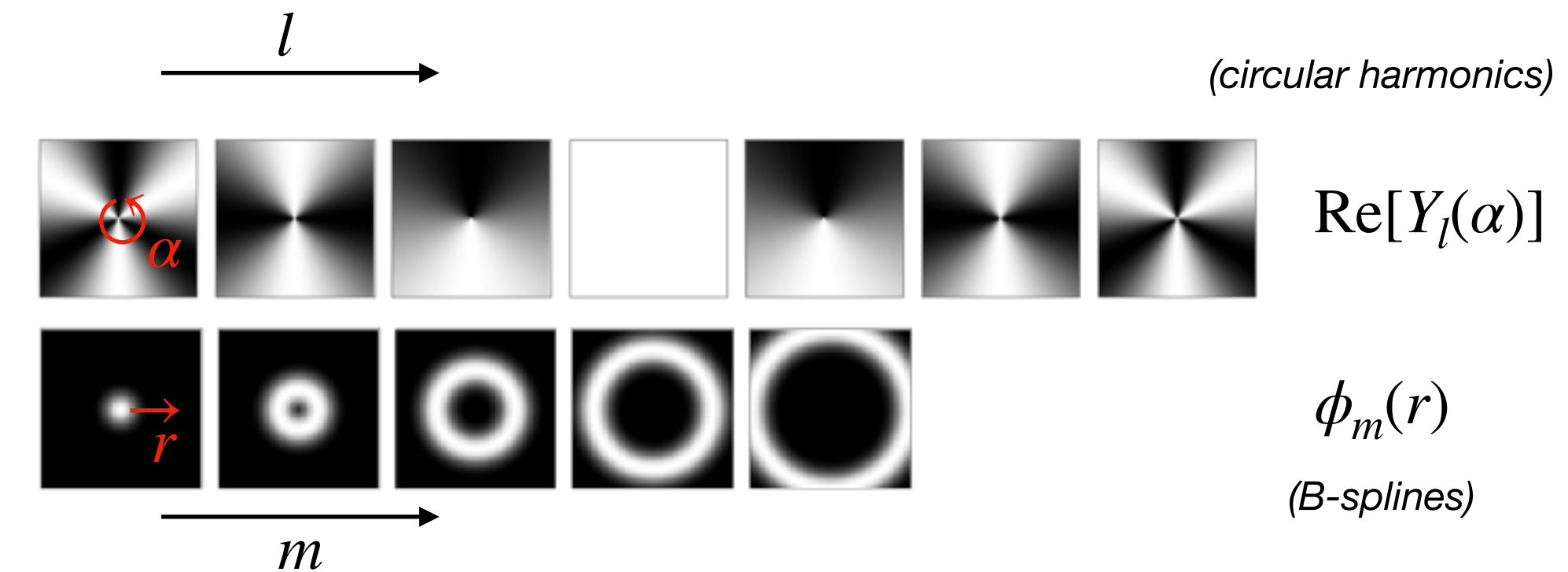
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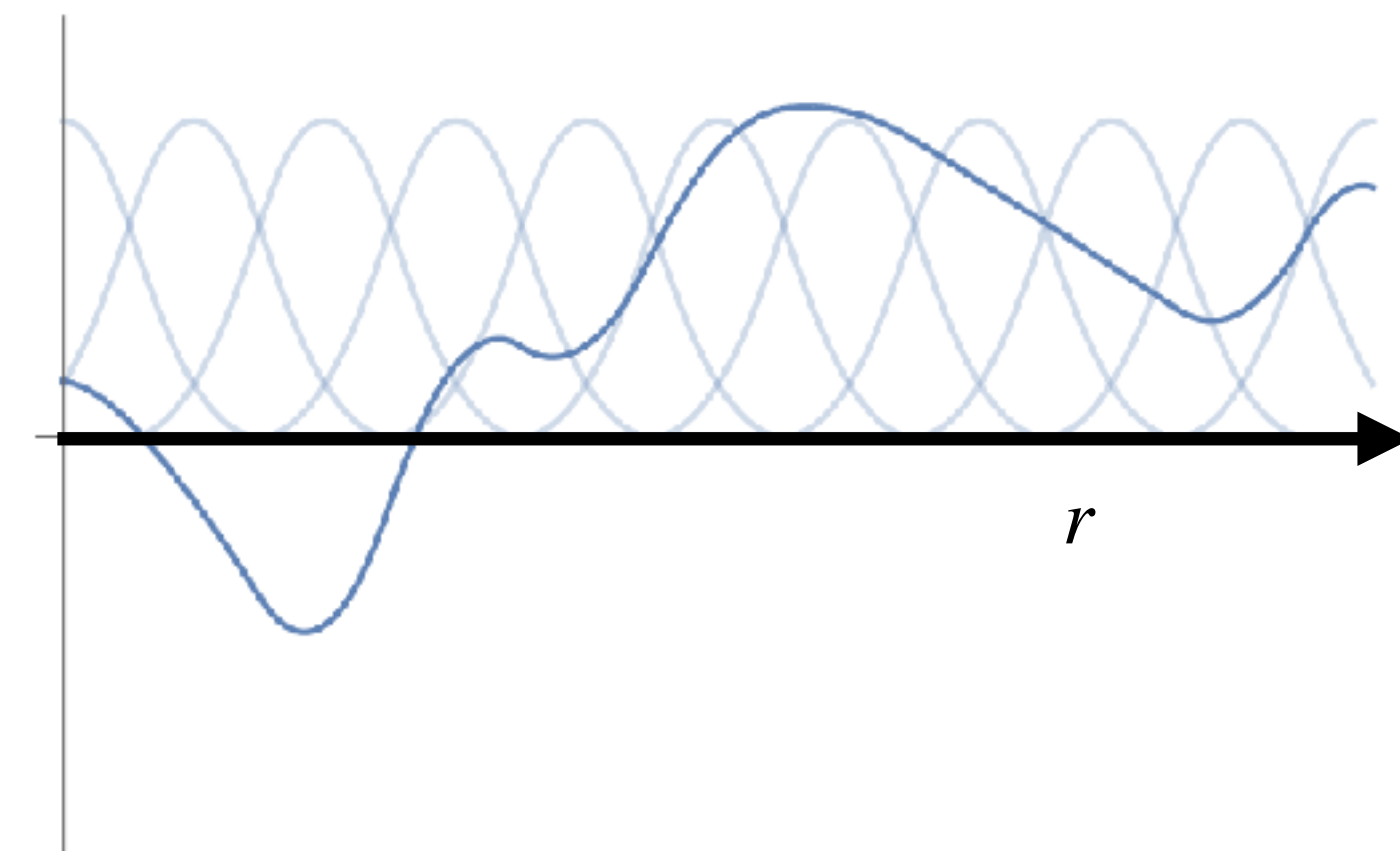
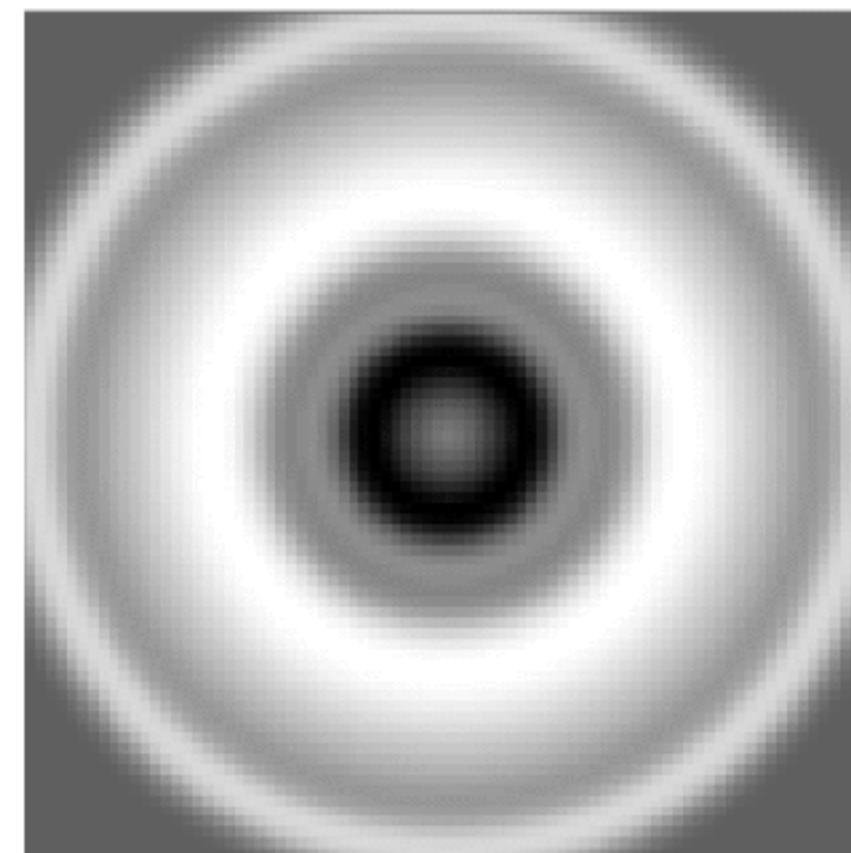
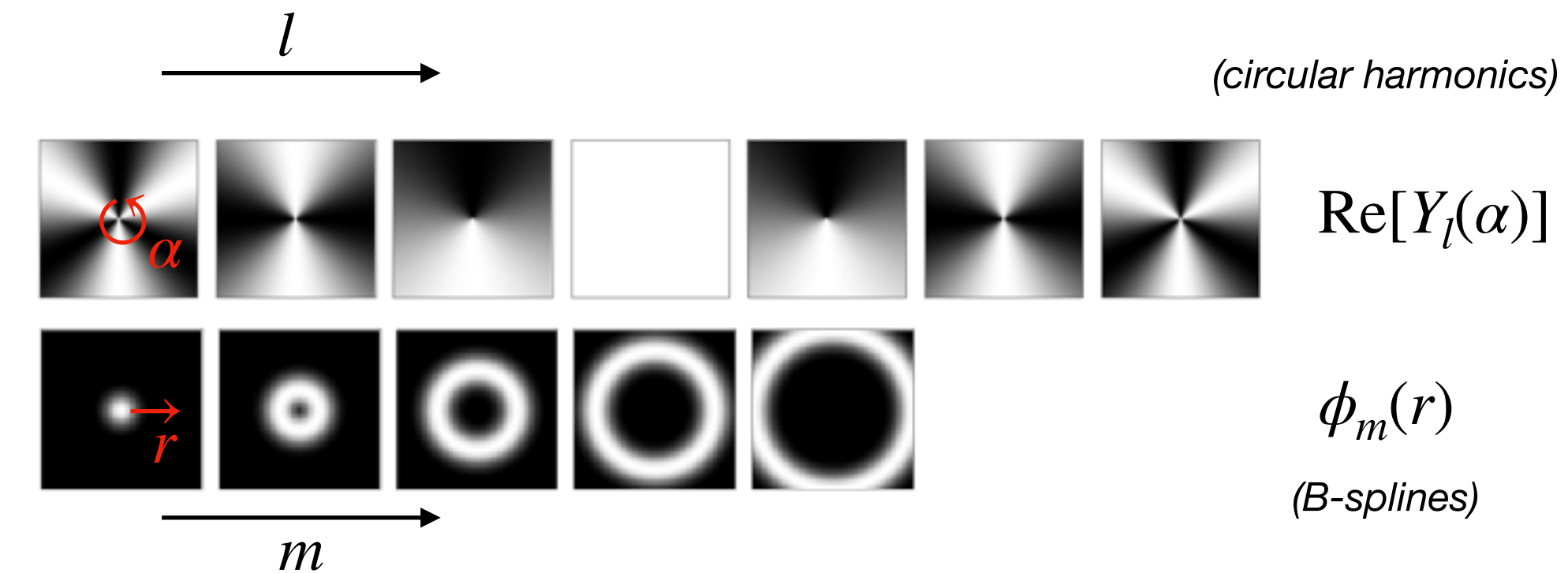
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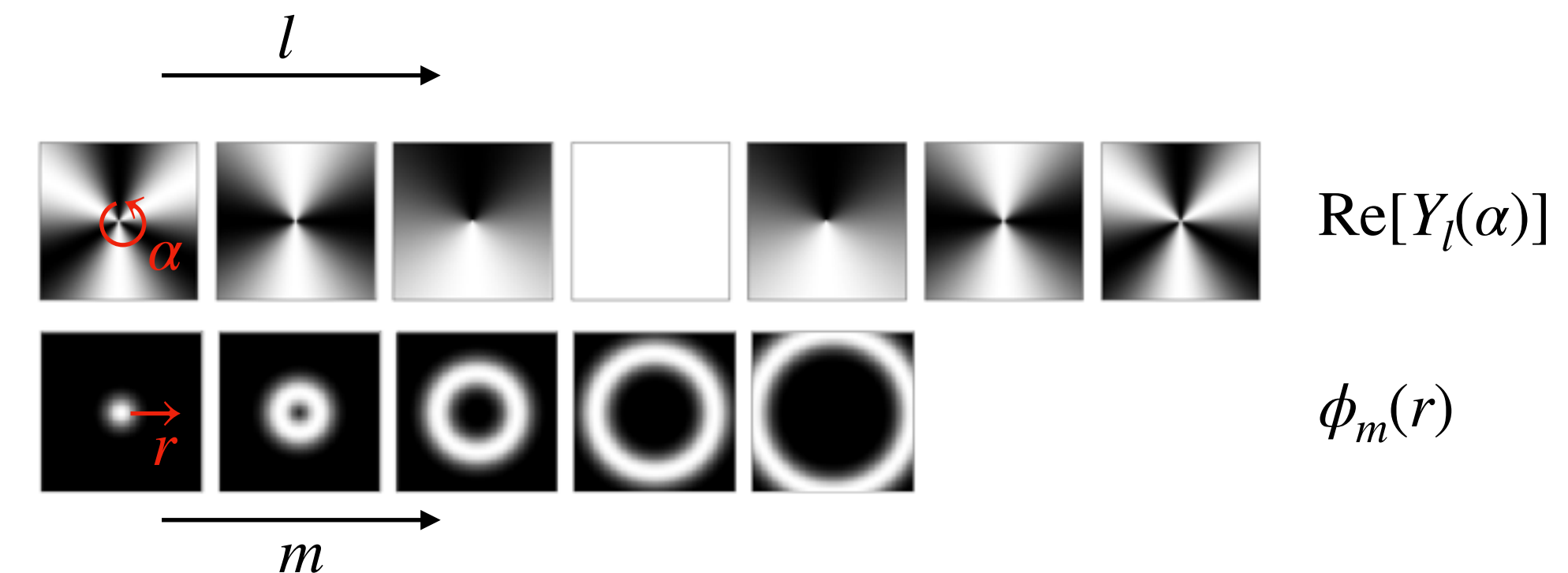
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$$\begin{aligned} k(\mathbf{x} | \mathbf{w}) &= \sum_l \sum_m w_m \bar{w}_l \phi_m(r) Y_l(\alpha) \\ &= \sum_l \sum_m \bar{w}_{ml} \phi_m(r) Y_l(\alpha) && \text{("absorb" weights)} \\ &= \sum_l \hat{w}_l(r) Y_l(\alpha) && \text{with radius dependent weights } \hat{w}_l(r) = \sum_m w_{ml} \phi_m(r) \end{aligned}$$

- Then such kernel is clearly rotation steerable!

$$k(\mathbf{R}_{\theta}^{-1} \mathbf{x} | \hat{\mathbf{w}}(r)) = k(\mathbf{x} | \rho(\theta) \hat{\mathbf{w}}(r))$$



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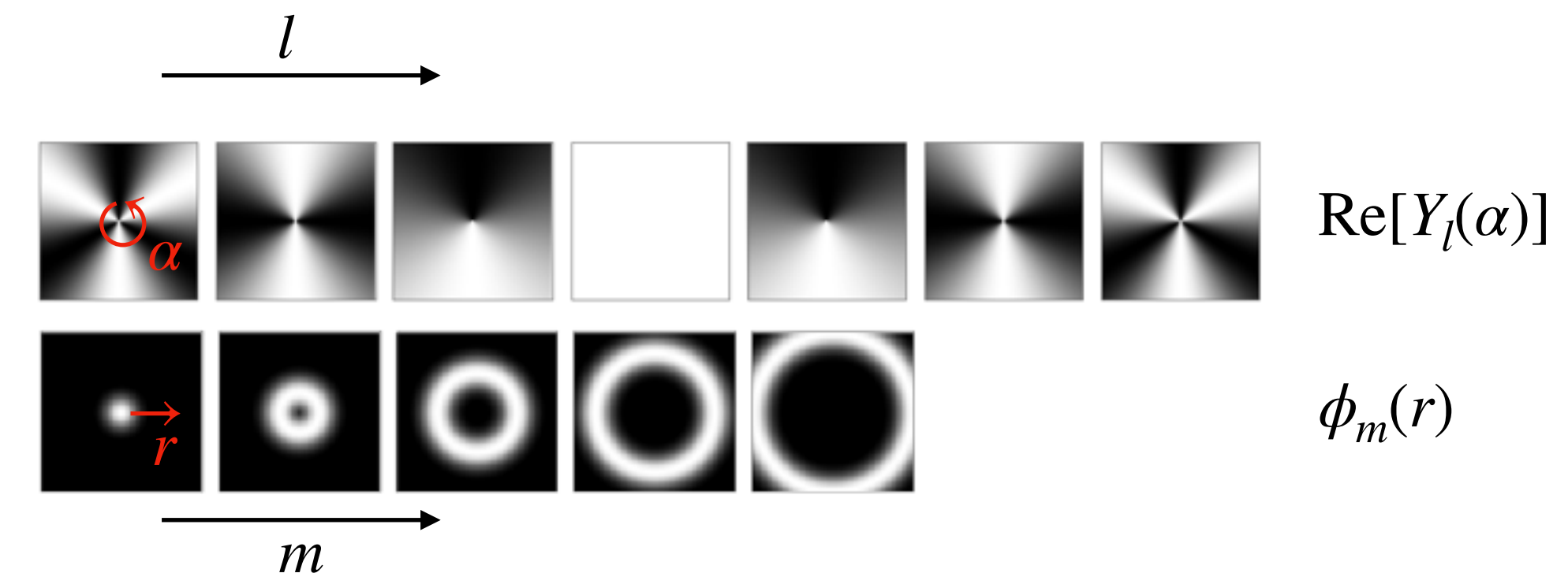
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with radius dependent weights $\hat{w}_l(r) = \sum_m \bar{w}_{ml} \phi_m(r)$

Or directly parametrize as $\hat{\mathbf{w}}(r) = \text{MLP}(r | \mathbf{w})$!

Complex (irreducible) representations

$$\begin{array}{c}
 Y(\mathbf{R}_\theta^{-1} \mathbf{x}) \\
 \begin{array}{cc} \text{Re} & \text{Im} \end{array} \\
 \left(\begin{array}{cc} \begin{array}{c} \text{[Image 1]}\end{array} & \begin{array}{c} \text{[Image 2]}\end{array} \\ \begin{array}{c} \text{[Image 3]}\end{array} & \begin{array}{c} \text{[Image 4]}\end{array} \\ \begin{array}{c} \text{[Image 5]}\end{array} & \begin{array}{c} \text{[Image 6]}\end{array} \\ \rightarrow & \rightarrow \\ \begin{array}{c} \text{[Image 7]}\end{array} & \begin{array}{c} \text{[Image 8]}\end{array} \\ \begin{array}{c} \text{[Image 9]}\end{array} & \begin{array}{c} \text{[Image 10]}\end{array} \\ \begin{array}{c} \text{[Image 11]}\end{array} & \begin{array}{c} \text{[Image 12]}\end{array} \end{array} \right)
 \end{array}
 =
 \begin{array}{c}
 \rho(\mathbf{R}_\theta^{-1}) \\
 \begin{pmatrix}
 e^{3i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & e^{1i\theta} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta}
 \end{pmatrix}
 \end{array}
 \begin{array}{c}
 Y(\mathbf{x}) \\
 \begin{array}{cc} \text{Re} & \text{Im} \end{array} \\
 \left(\begin{array}{cc} \begin{array}{c} \text{[Image 13]}\end{array} & \begin{array}{c} \text{[Image 14]}\end{array} \\ \begin{array}{c} \text{[Image 15]}\end{array} & \begin{array}{c} \text{[Image 16]}\end{array} \\ \begin{array}{c} \text{[Image 17]}\end{array} & \begin{array}{c} \text{[Image 18]}\end{array} \\ \rightarrow & \rightarrow \\ \begin{array}{c} \text{[Image 19]}\end{array} & \begin{array}{c} \text{[Image 20]}\end{array} \\ \begin{array}{c} \text{[Image 21]}\end{array} & \begin{array}{c} \text{[Image 22]}\end{array} \\ \begin{array}{c} \text{[Image 23]}\end{array} & \begin{array}{c} \text{[Image 24]}\end{array} \end{array} \right)
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 Y(\mathbf{R}_\theta^{-1} \mathbf{x}) \\
 \begin{array}{cc} \text{Re} & \text{Im} \end{array} \\
 \left(\begin{array}{cc} \begin{array}{c} \text{[Image 1]} \\ \text{[Image 2]} \\ \text{[Image 3]} \\ \rightarrow \\ \text{[Image 4]} \\ \text{[Image 5]} \\ \text{[Image 6]} \end{array} & \begin{array}{c} \text{[Image 1]} \\ \text{[Image 2]} \\ \text{[Image 3]} \\ \rightarrow \\ \text{[Image 4]} \\ \text{[Image 5]} \\ \text{[Image 6]} \end{array} \end{array} \right)
 \end{array}
 =
 \begin{array}{c}
 \rho(\mathbf{R}_\theta^{-1}) \\
 \left(\begin{array}{ccccccc}
 e^{3i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & e^{1i\theta} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta}
 \end{array} \right)
 \end{array}
 \begin{array}{c}
 Y(\mathbf{x}) \\
 \begin{array}{cc} \text{Re} & \text{Im} \end{array} \\
 \left(\begin{array}{cc} \begin{array}{c} \text{[Image 1]} \\ \text{[Image 2]} \\ \text{[Image 3]} \\ \rightarrow \\ \text{[Image 4]} \\ \text{[Image 5]} \\ \text{[Image 6]} \end{array} & \begin{array}{c} \text{[Image 1]} \\ \text{[Image 2]} \\ \text{[Image 3]} \\ \rightarrow \\ \text{[Image 4]} \\ \text{[Image 5]} \\ \text{[Image 6]} \end{array} \end{array} \right)
 \end{array}$$

Complex (irreducible) representations

$$\begin{array}{c}
 Y(\mathbf{R}_\theta^{-1} \mathbf{x}) \\
 \begin{array}{cc} \text{Re} & \text{Im} \end{array} \\
 \left(\begin{array}{c} \text{[6 images]} \end{array} \right) \\
 \begin{array}{cc} \longrightarrow & \longrightarrow \end{array} \\
 \left(\begin{array}{c} \text{[6 images]} \end{array} \right)
 \end{array}
 =
 \begin{array}{c}
 \rho(\mathbf{R}_\theta^{-1}) \\
 \left(\begin{array}{cccccc}
 e^{3i\theta} & 0 & 0 & 0 & 0 & 0 \\
 0 & e^{2i\theta} & 0 & 0 & 0 & 0 \\
 0 & 0 & e^{1i\theta} & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 \\
 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} \\
 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta}
 \end{array} \right)
 \end{array}
 \begin{array}{c}
 Y(\mathbf{x}) \\
 \begin{array}{cc} \text{Re} & \text{Im} \end{array} \\
 \left(\begin{array}{c} \text{[6 images]} \end{array} \right) \\
 \begin{array}{cc} \longrightarrow & \longrightarrow \end{array} \\
 \left(\begin{array}{c} \text{[6 images]} \end{array} \right)
 \end{array}$$

$\cos(l\alpha)$ $\sin(l\alpha)$

Real (irreducible) representations

$$Y(\mathbf{R}_\theta^{-1} \mathbf{x}) = \rho(\mathbf{R}_\theta^{-1}) Y(\mathbf{x})$$

$$\begin{pmatrix} \text{[white box]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 & 0 & 0 & 0 \\ 0 & -\sin \theta & \cos \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 2\theta & \sin 2\theta & 0 & 0 \\ 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\ 0 & 0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix} \begin{pmatrix} \text{[white box]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \\ \text{[quadrant pattern]} \end{pmatrix}$$

Real (irreducible) representations

$$Y(\mathbf{R}_\theta^{-1} \mathbf{x}) = \rho(\mathbf{R}_\theta^{-1}) Y(\mathbf{x})$$

$$\begin{pmatrix} \text{[white box]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 & 0 & 0 & 0 \\ 0 & -\sin \theta & \cos \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 2\theta & \sin 2\theta & 0 & 0 \\ 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\ 0 & 0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix} \begin{pmatrix} \text{[white box]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \\ \text{[square wave]} \end{pmatrix}$$

Real (irreducible) representations

The real basis functions $Y_l(\mathbf{x}) = \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix}$ are steerable using $\rho_l(\mathbf{R}_\theta) = \begin{pmatrix} \cos l\theta & -\sin l\theta \\ \sin l\theta & \cos l\theta \end{pmatrix}$

Proof:

$$\begin{aligned} Y_l(\mathbf{R}_\theta^{-1}\mathbf{x}) &= \begin{pmatrix} \cos(l(\alpha - \theta)) \\ \sin(l(\alpha - \theta)) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\textcolor{blue}{l}\alpha + \textcolor{red}{-}l\theta) \\ \sin(\textcolor{blue}{l}\alpha + \textcolor{red}{-}l\theta) \end{pmatrix} = \begin{pmatrix} \cos(\textcolor{blue}{l}\alpha)\cos(\textcolor{red}{-}l\theta) - \sin(\textcolor{blue}{l}\alpha)\sin(\textcolor{red}{-}l\theta) \\ \sin(\textcolor{blue}{l}\alpha)\cos(\textcolor{red}{-}l\theta) + \cos(\textcolor{blue}{l}\alpha)\sin(\textcolor{red}{-}l\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos\textcolor{red}{-}l\theta & -\sin\textcolor{red}{-}l\theta \\ \sin\textcolor{red}{-}l\theta & \cos\textcolor{red}{-}l\theta \end{pmatrix} \begin{pmatrix} \cos(\textcolor{blue}{l}\alpha) \\ \sin(\textcolor{blue}{l}\alpha) \end{pmatrix} \\ &= \rho_l(\mathbf{R}_\theta^{-1}) Y_l(\mathbf{x}) \end{aligned}$$

$$\begin{pmatrix} \begin{matrix} \text{cosine lobe} \\ \text{sinusoidal lobe} \\ \text{sinusoidal lobe} \end{matrix} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\ 0 & 0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix} \begin{pmatrix} \text{cosine lobe} \\ \text{sinusoidal lobe} \\ \text{sinusoidal lobe} \end{pmatrix}$$

Real (irreducible) representations

The real basis functions $Y_l(\mathbf{x}) = \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix}$ are steerable using $\rho_l(\mathbf{R}_\theta) = \begin{pmatrix} \cos l\theta & -\sin l\theta \\ \sin l\theta & \cos l\theta \end{pmatrix}$

Proof:

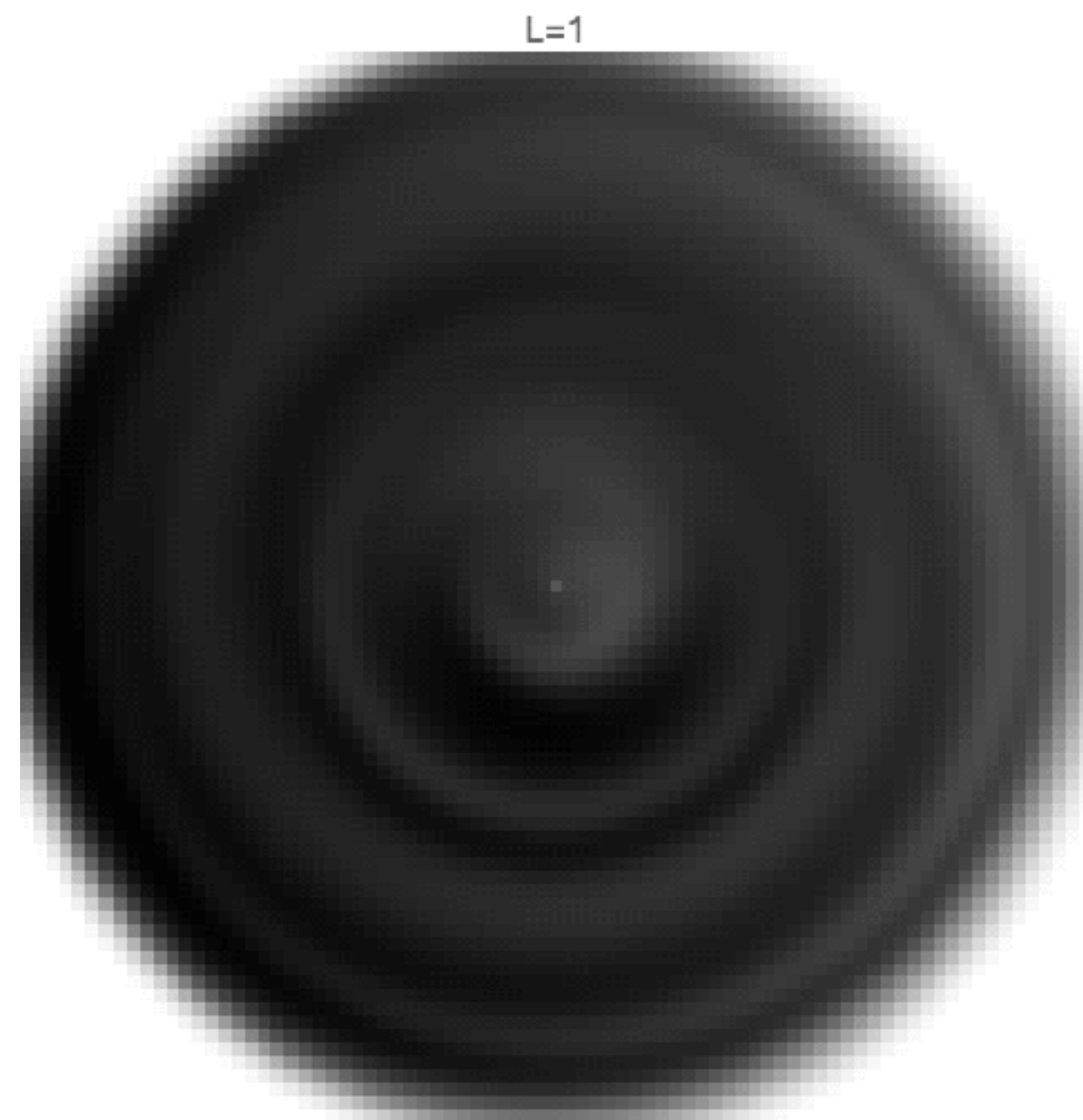
$$\begin{aligned} Y_l(\mathbf{R}_\theta^{-1}\mathbf{x}) &= \begin{pmatrix} \cos(l(\alpha - \theta)) \\ \sin(l(\alpha - \theta)) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\textcolor{blue}{l}\alpha + \textcolor{red}{-}l\theta) \\ \sin(\textcolor{blue}{l}\alpha + \textcolor{red}{-}l\theta) \end{pmatrix} = \begin{pmatrix} \cos(\textcolor{blue}{l}\alpha)\cos(\textcolor{red}{-}l\theta) - \sin(\textcolor{blue}{l}\alpha)\sin(\textcolor{red}{-}l\theta) \\ \sin(\textcolor{blue}{l}\alpha)\cos(\textcolor{red}{-}l\theta) + \cos(\textcolor{blue}{l}\alpha)\sin(\textcolor{red}{-}l\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos\textcolor{red}{-}l\theta & -\sin\textcolor{red}{-}l\theta \\ \sin\textcolor{red}{-}l\theta & \cos\textcolor{red}{-}l\theta \end{pmatrix} \begin{pmatrix} \cos(\textcolor{blue}{l}\alpha) \\ \sin(\textcolor{blue}{l}\alpha) \end{pmatrix} \\ &= \rho_l(\mathbf{R}_\theta^{-1}) Y_l(\mathbf{x}) \end{aligned}$$

$$\begin{pmatrix} \begin{matrix} \text{cosine lobe} \\ \text{sinusoidal lobe} \\ \text{sinusoidal lobe} \end{matrix} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\ 0 & 0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix} \begin{pmatrix} \text{cosine lobe} \\ \text{sinusoidal lobe} \\ \text{sinusoidal lobe} \end{pmatrix}$$

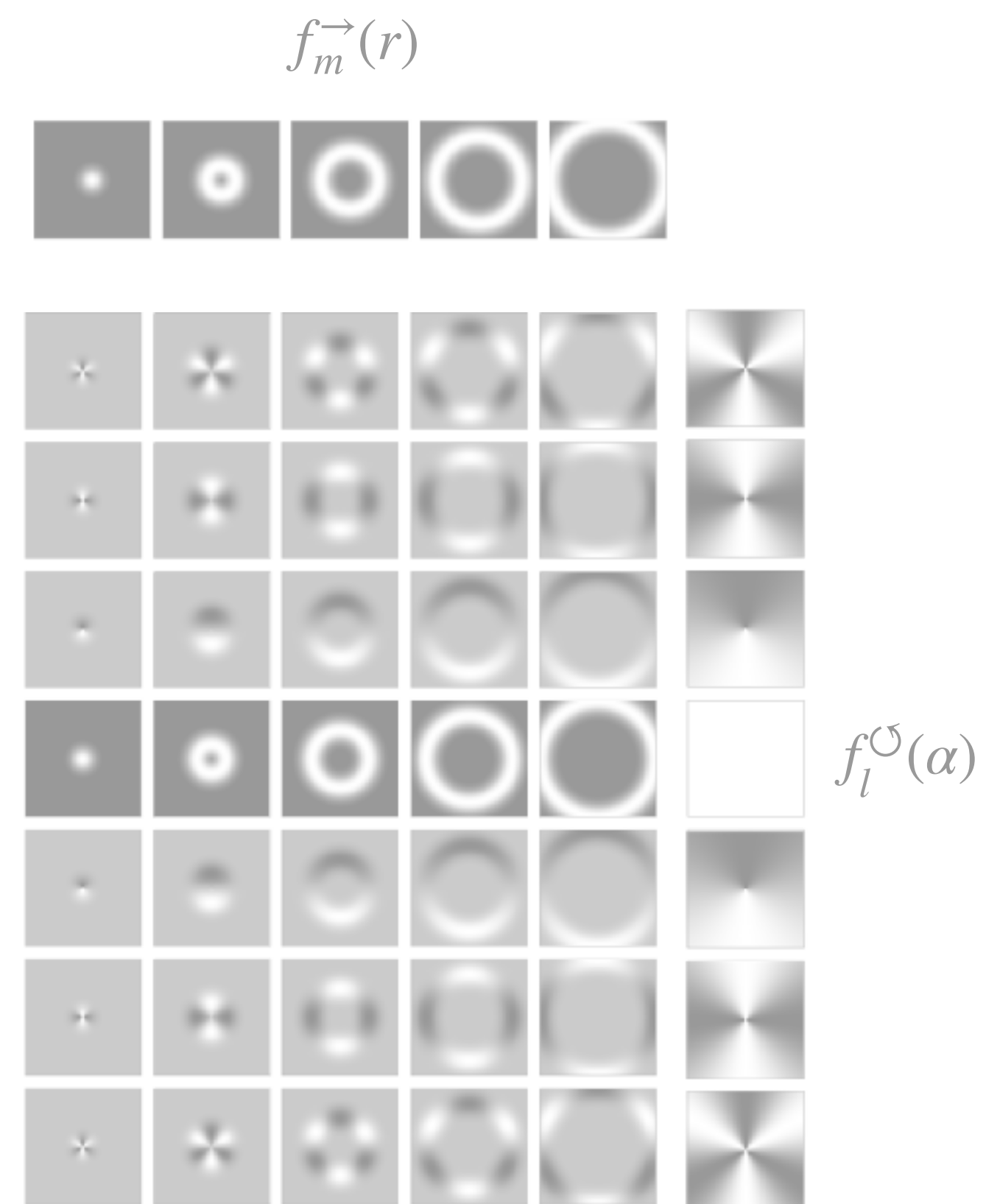
Representing interesting convolution kernels in a steerable basis!

Exercise:

1. Tune the weights $\hat{\mathbf{w}}$ until you get something interesting.
2. Add more detail by increasing maximum frequency!



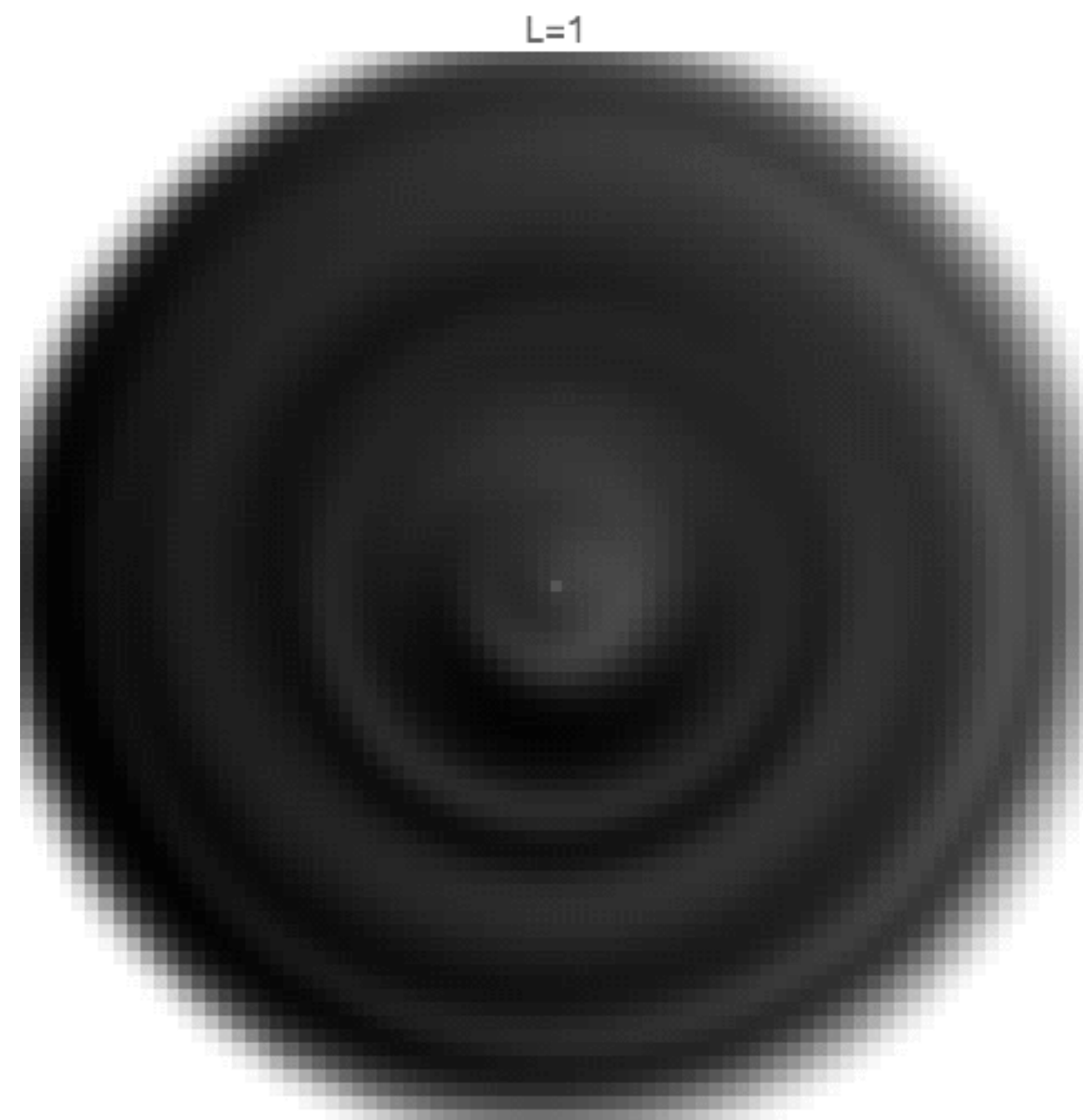
$$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$$



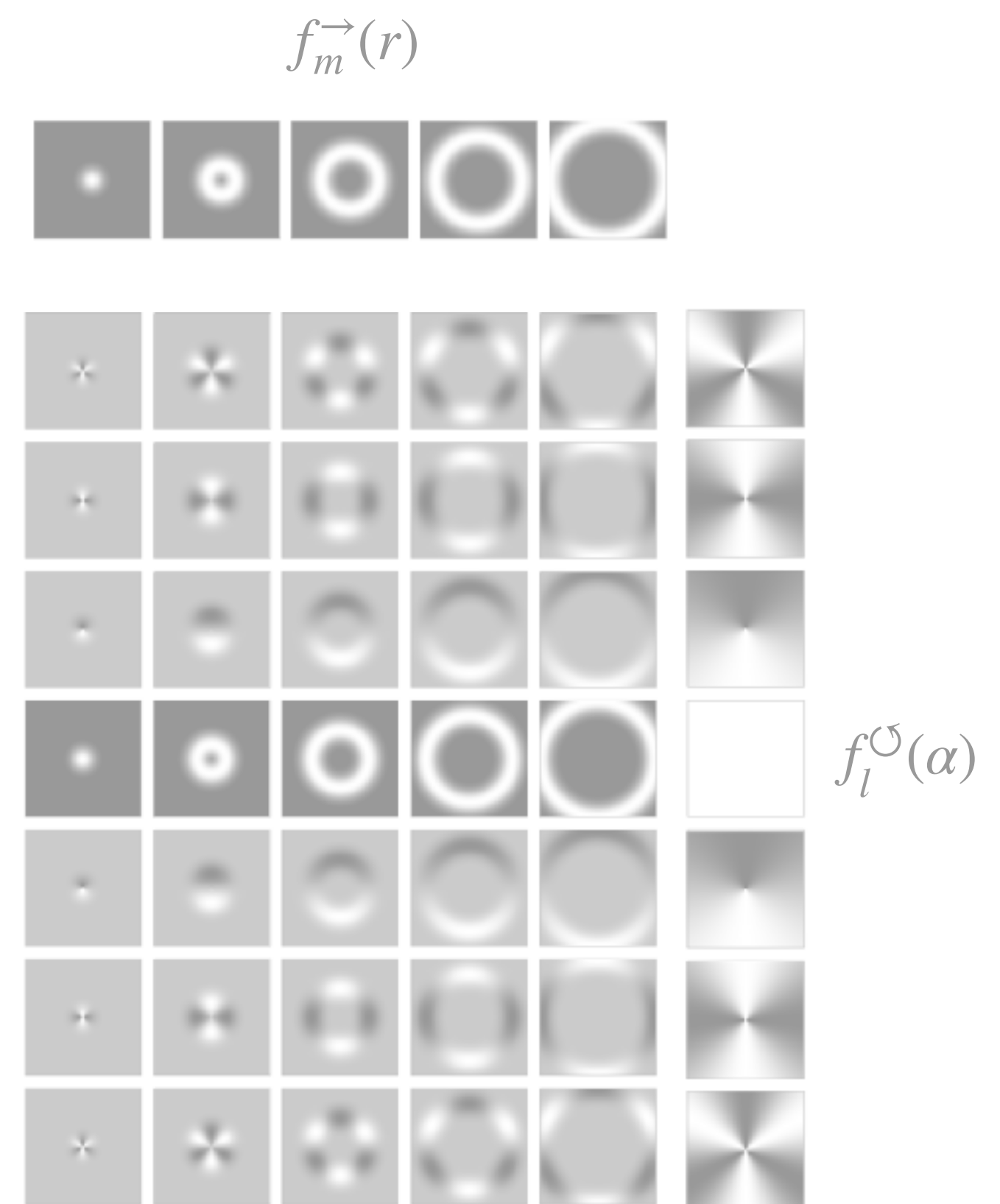
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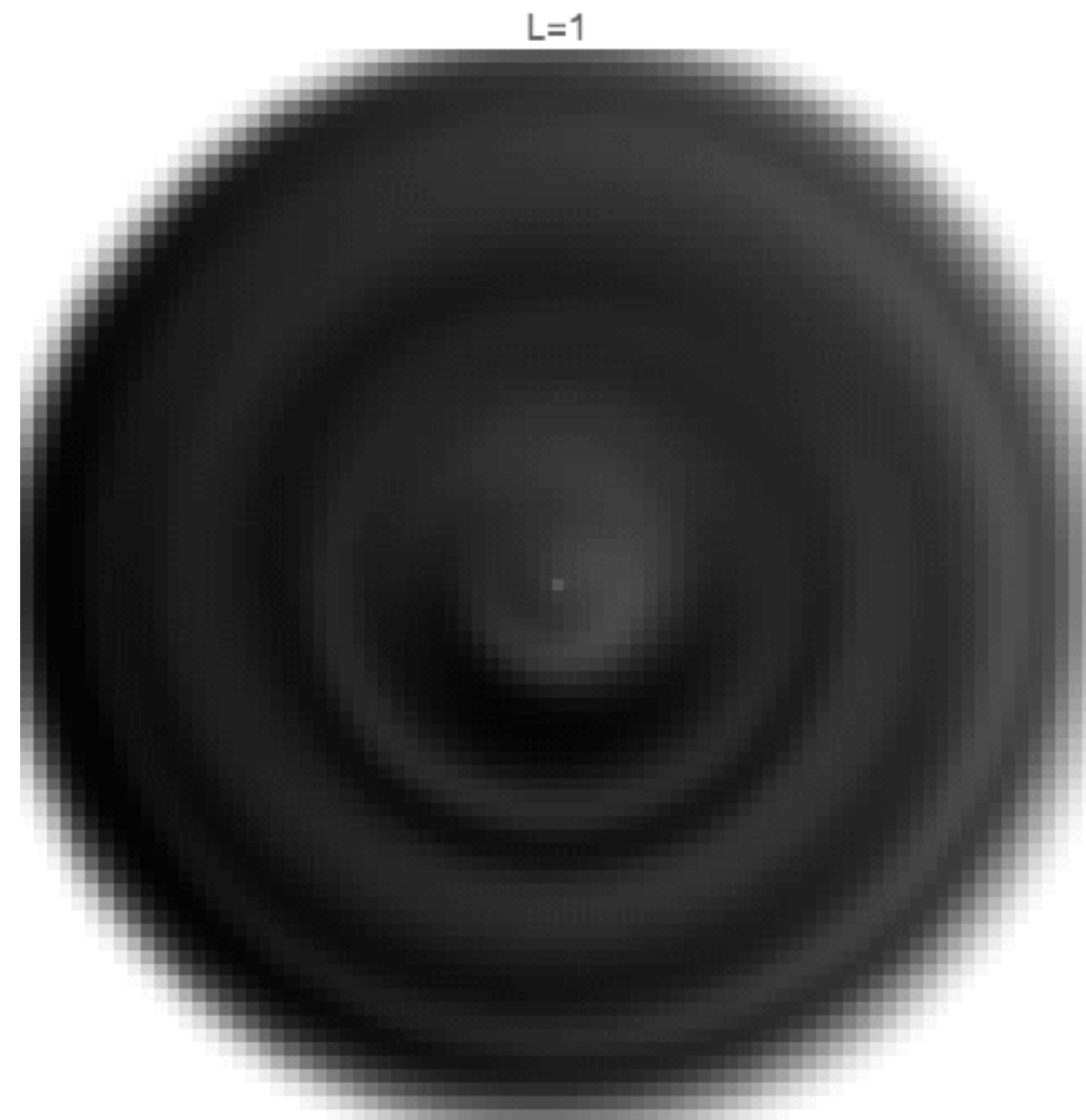
$$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$$



Representing interesting convolution kernels in a steerable basis!

Exercise:

1. Tune the weights $\hat{\mathbf{w}}$ until you get something interesting.
2. Add more detail by increasing maximum frequency!



$$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$$

3. Go crazy and **steer** it by transforming the weights!



$$k(\mathbf{x} \mid \rho(\theta)\hat{\mathbf{w}}(r))$$

