

# Group Equivariant Deep Learning

## Lecture 2 - Steerable group convolutions

### Lecture 2.1 - Steerable kernels/basis functions

*Definition and  $SO(2)$  example*

Erik Bekkers, Amsterdam Machine Learning Lab, University of Amsterdam

This mini-course serves as a module with the UvA Master AI course Deep Learning 2 <https://uvadl2c.github.io/>

# Group Equivariant Deep Learning

## Lecture 2 - Steerable group convolutions

### Lecture 2.1 - Steerable kernels/basis functions

*Definition and  $SO(2)$  example*

### Lecture 2.2 - Revisiting regular G-convs with steerable kernels | Template matching viewpoint

*Motivating the Fourier transform on  $G$  and showing we now no longer need a grid on the sub-group  $H$ !*

### Lecture 2.3 - Group Theory | Irreducible representations and Fourier transform

*Preliminaries for steerable feature fields and steerable g-conv intuition with a focus on  $SO(2)$*

### Lecture 2.4 - Group Theory | Induced representations and feature fields

*Preliminaries (and intuition) for steerable group convolutions*

### Lecture 2.5 - Steerable group convolutions

*And how to use them*

### Lecture 2.6 - Activation functions for steerable G-CNNs

*Examples of which we can and cannot use*

### Lecture 2.7 - Derivation of Harmonic networks<sup>1</sup> from regular g-convs | Recalling g-convs are all you need!

<sup>1</sup>Worrall, D. E., Garbin, S. J., Turmukhambetov, D., & Brostow, G. J. Harmonic networks: Deep translation and rotation equivariance. CVPR 2017

# Steerable basis

A vector  $Y(x) = \begin{pmatrix} \vdots \\ Y_l(x) \\ \vdots \end{pmatrix} \in \mathbb{K}^L$  with (basis) functions  $Y_l \in \mathbb{L}_2(X)$  is steerable if

$$\forall_{g \in G} : \quad Y(gx) = \rho(g)Y(x),$$

where  $gx$  denotes the action of  $G$  on  $X$  and  $\rho(g) \in \mathbb{K}^{L \times L}$  is a representation of  $G$ .

*I.e., we can transform all basis functions simply by taking a linear combination of the original basis functions.*

# Example: Steerable basis on $S^1$ (circular harmonics)

Basis functions (for  $\mathbb{L}_2(S^1)$ ):

$$Y_l(\alpha) = e^{il\alpha}$$

Are steered by representations:

$$\rho_l(\theta) = e^{il\theta}$$

Proof: 
$$\begin{aligned} Y_l(\alpha - \theta) &= e^{il(\alpha - \theta)} \\ &= e^{-il\theta} e^{il\alpha} \\ &= \rho_l(-\theta) Y_l(\alpha) \end{aligned}$$

# Example: Steerable basis on $S^1$ (circular harmonics)

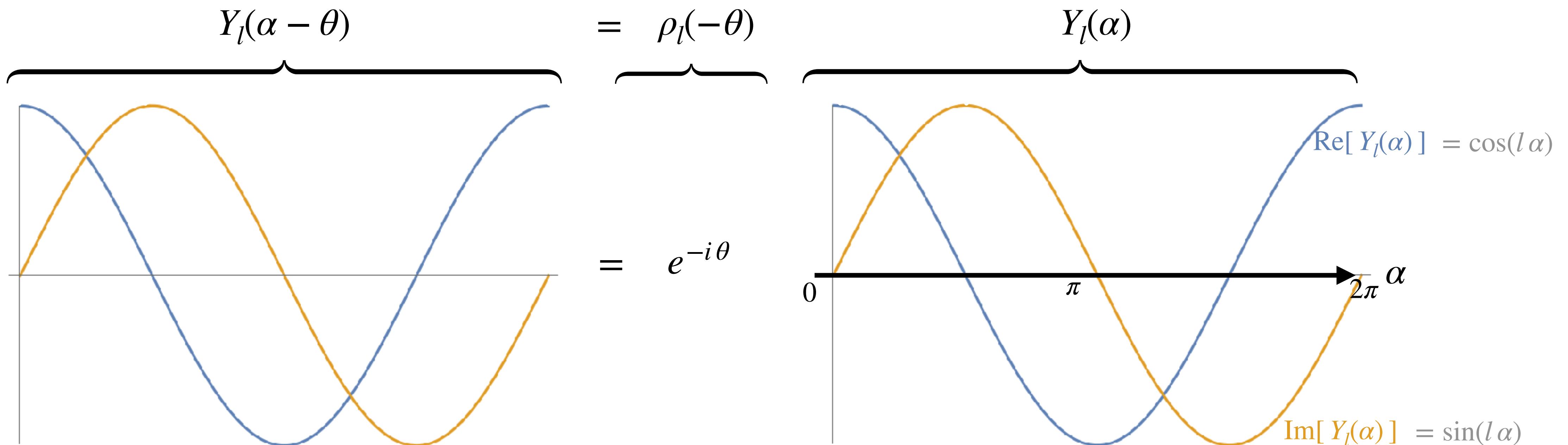
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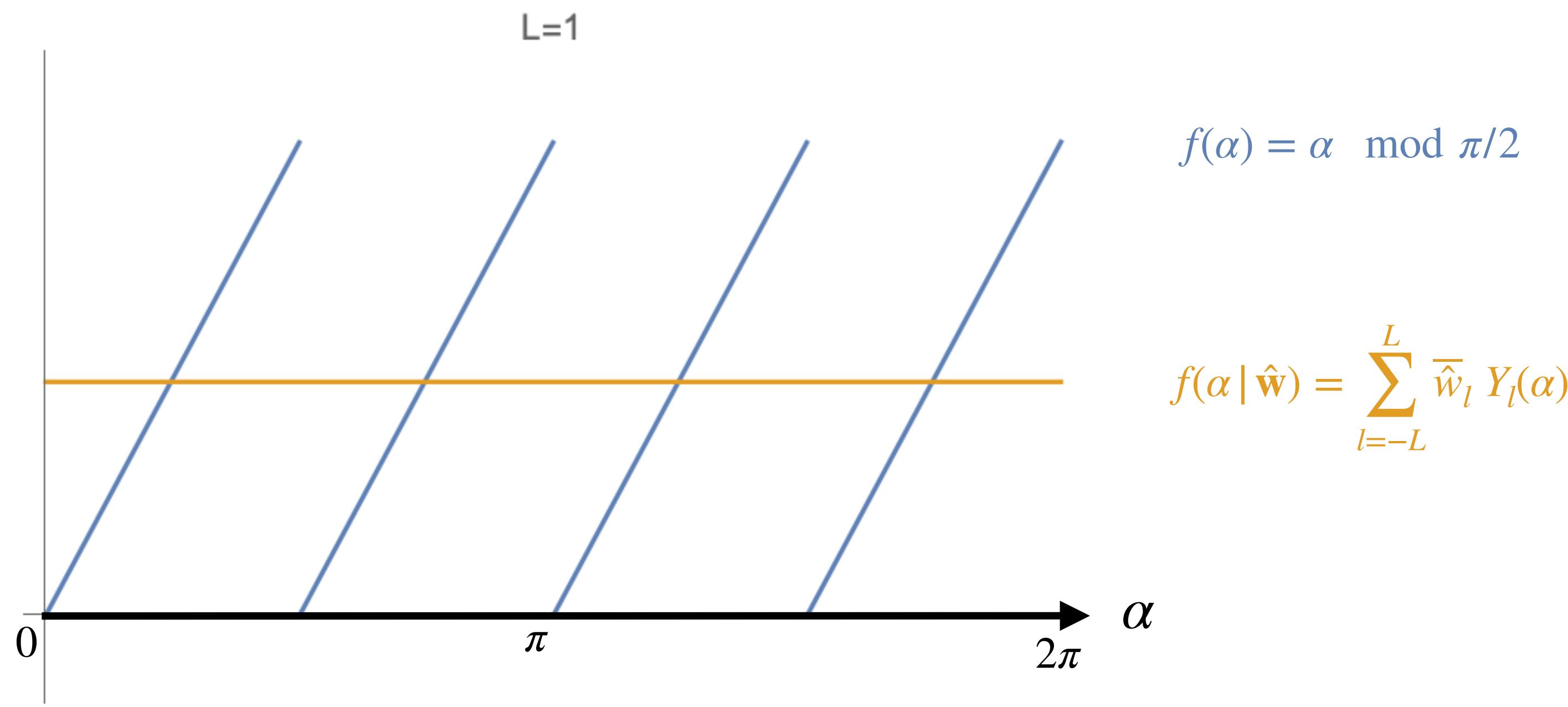
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**Form a complete orthonormal (Fourier) basis:**

$$f(\alpha | \hat{\mathbf{w}}) = \sum_{l=-\infty}^{\infty} \overline{\hat{w}_l} Y_l(\alpha)$$

$Y_l$  are given by the irreps of  $SO(2)$  and hence form orthogonal basis (Peter-Weyl Theorem)



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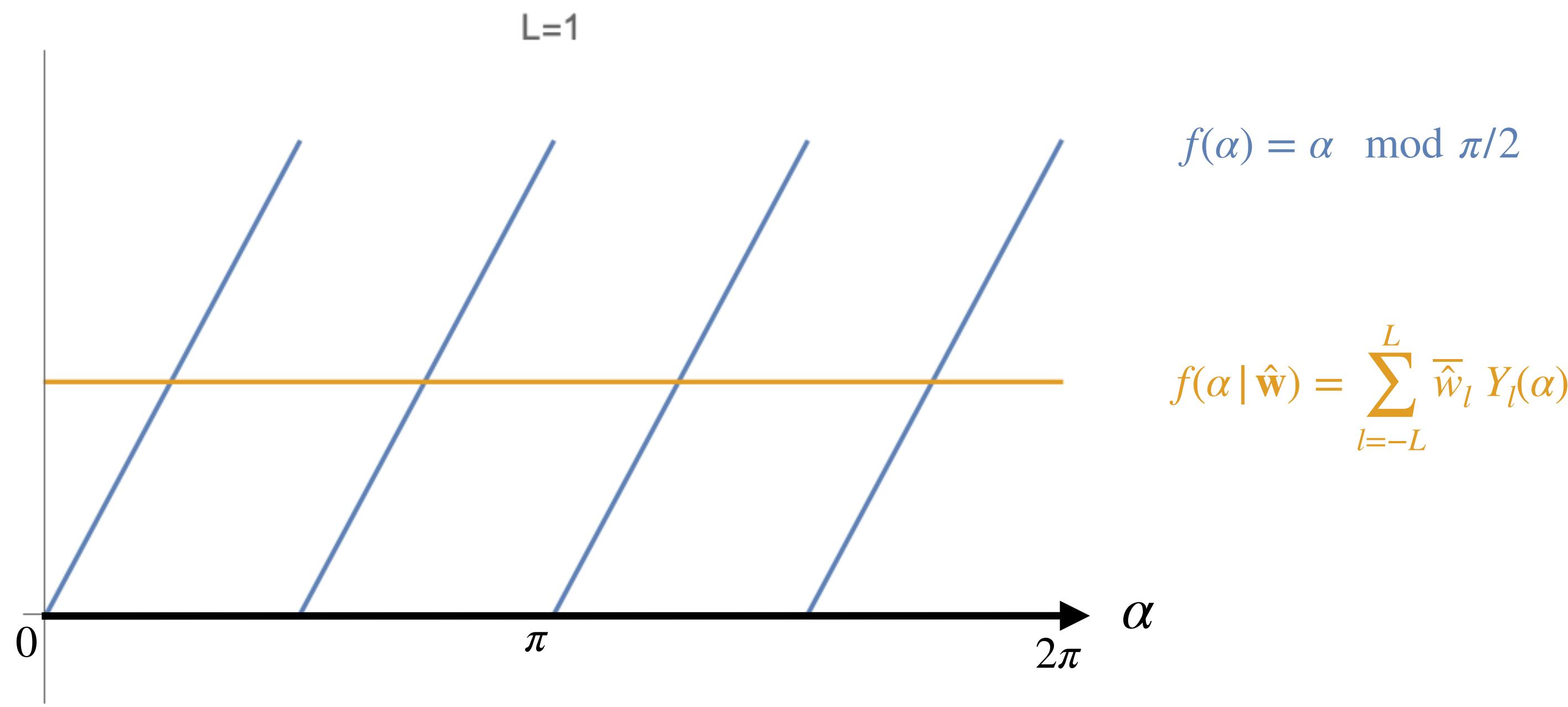
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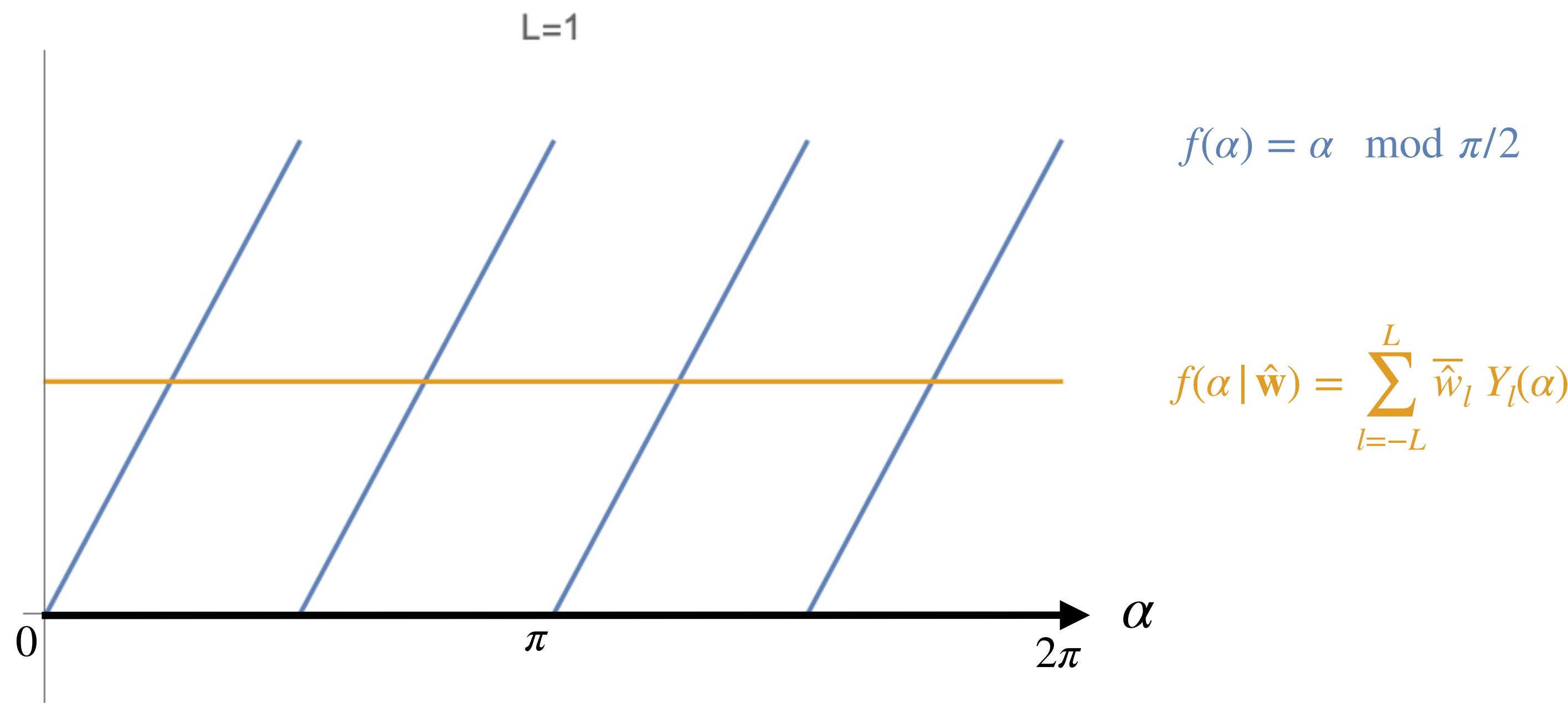
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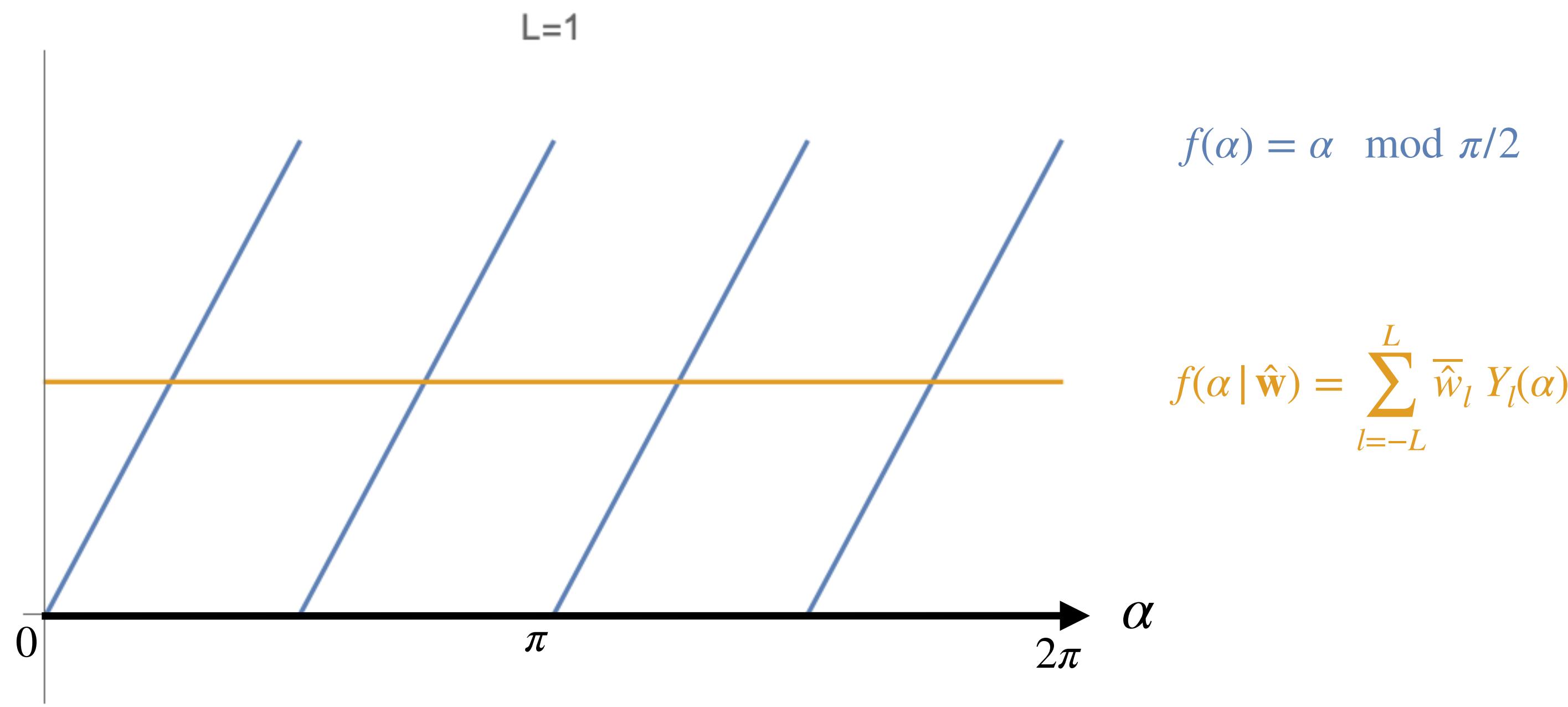
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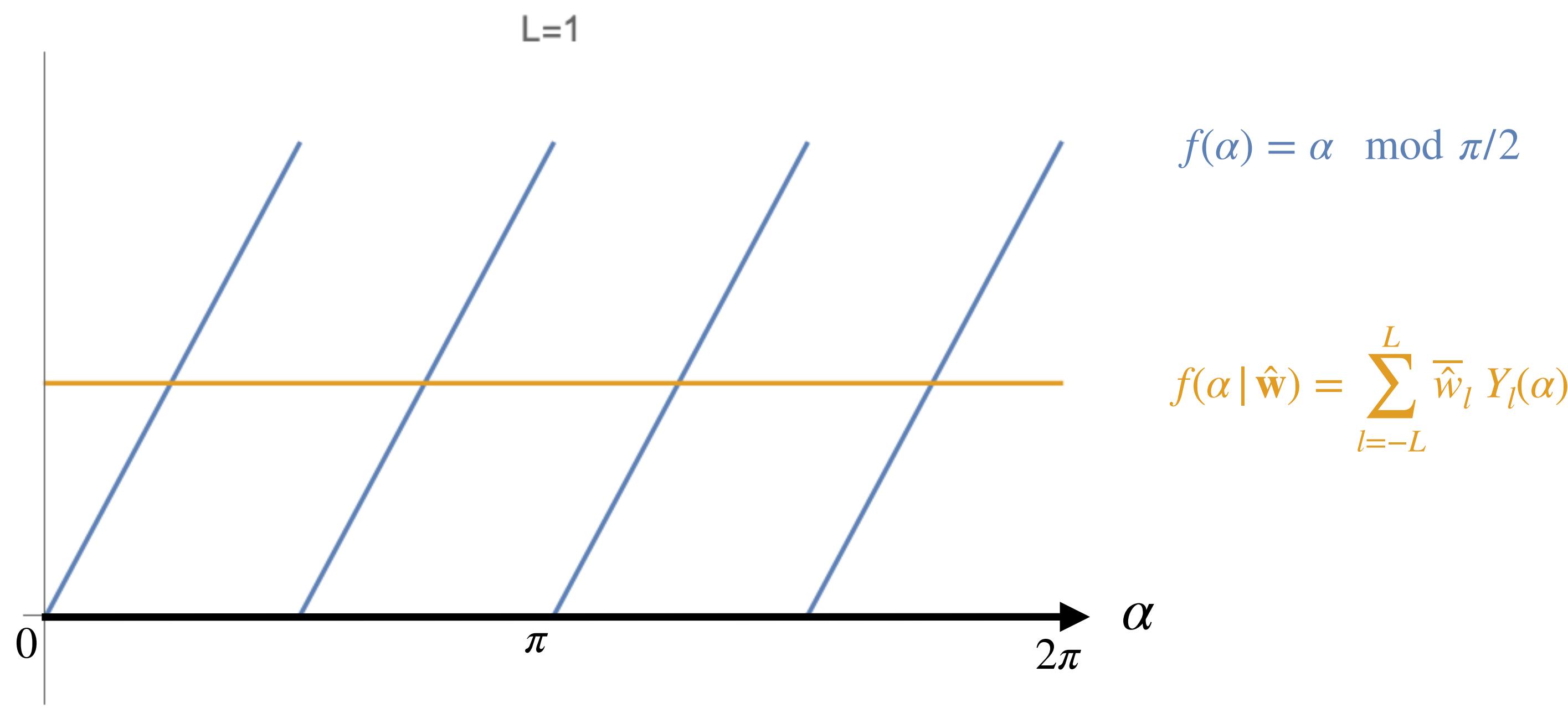
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# Example: Steerable basis on $S^1$ (circular harmonics)

$$\begin{array}{c}
 Y(\alpha - \theta) \\
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 = \left( \begin{array}{cccccc} e^{i3\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i2\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i1\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i1\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i2\theta} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} \text{blue line} \\ \text{orange line} \\ \text{blue line} \\ \text{orange line} \\ \text{blue line} \\ \text{orange line} \\ \text{blue line} \\ \text{orange line} \end{array} \right) \\
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$\rho(-\theta) = \bigoplus_{l=-L}^L \rho_l(-\theta)$

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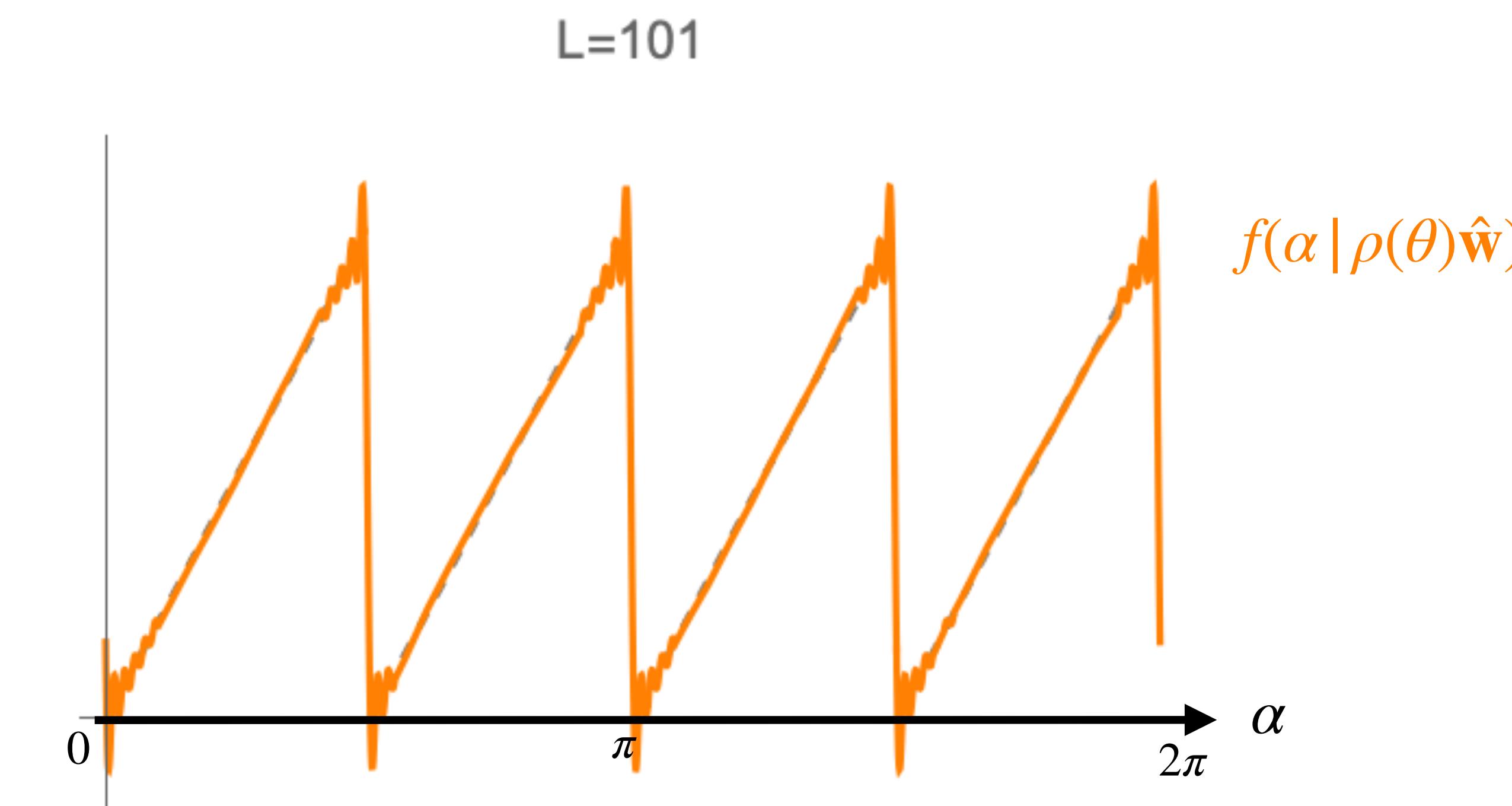
$$f(\alpha | \hat{\mathbf{w}}) = \hat{\mathbf{w}}^\dagger Y(\alpha)$$

Then we can **steer**/shift this function by transforming the weights  $\hat{\mathbf{w}}$

$$f(\alpha - \theta | \hat{\mathbf{w}}) = f(\alpha | \rho(\theta)\hat{\mathbf{w}})$$

Proof:

$$\begin{aligned} f(\alpha - \theta | \hat{\mathbf{w}}) &= \hat{\mathbf{w}}^\dagger Y(\alpha - \theta) \\ &= \hat{\mathbf{w}}^\dagger \rho(-\theta) Y(\alpha) \\ &= \hat{\mathbf{w}}^\dagger \rho(\theta)^\dagger Y(\alpha) \\ &= (\rho(\theta)\hat{\mathbf{w}})^\dagger Y(\alpha) \\ &= f(\alpha | \rho(\theta)\hat{\mathbf{w}}) \end{aligned}$$



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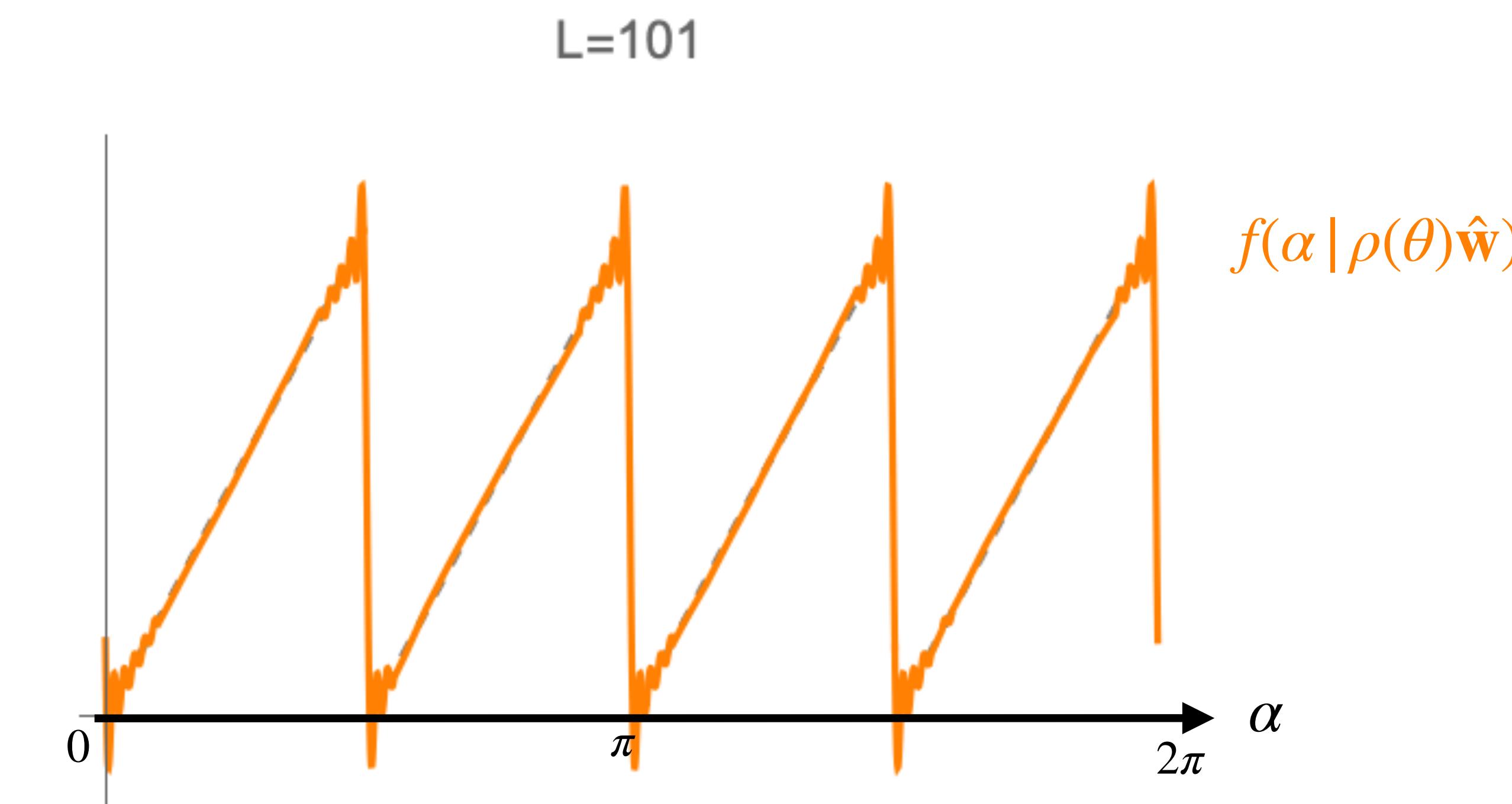
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# Two dimensional rotation-steerable functions

- The previous functions  $\rho_l(\theta) = e^{il\theta}$  are (irreducible) representations of  $SO(2)$

Recall lecture 1.6 (Group Theory | Homogeneous/quotient spaces)

## Transitive action

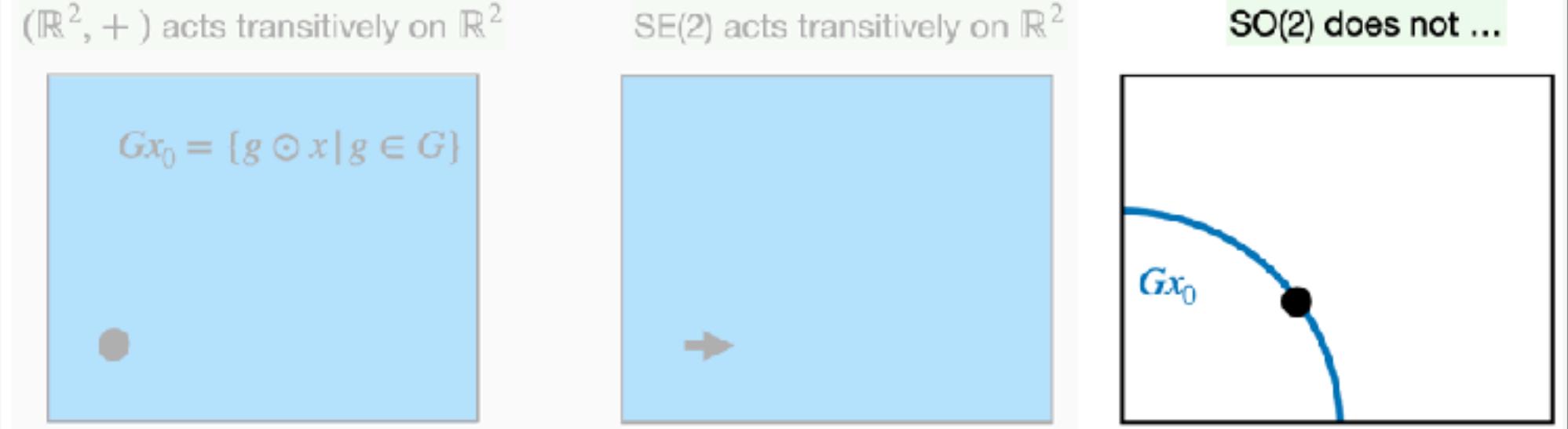
**Transitive action:** An action  $\odot : G \times X \rightarrow X$  such that

$$\forall_{x_0, x \in X} \exists_{g \in G} : x = g \odot x_0$$

$(\mathbb{R}^2, +)$  acts transitively on  $\mathbb{R}^2$

$SE(2)$  acts transitively on  $\mathbb{R}^2$

$SO(2)$  does not ...



6

- Though not transitively...
- It does act transitively on  $S^1$  though
- Use polar coordinates  $\mathbb{R}^2 \ni \mathbf{x} \leftrightarrow (r, \alpha) \in \mathbb{R}^+ \times S^1$  to come up with a rotation-steerable basis for  $\mathbb{L}_2(\mathbb{R}^2)$ !

# Two dimensional rotation-steerable functions

- Consider a function  $f(\mathbf{x}) = \bar{f}(r, \alpha)$  in polar coordinates

$$\mathbf{x} = (r \cos \alpha, r \sin \alpha)$$

- The action of  $SO(2)$  on  $\mathbb{R}^2$  in polar coords translates to

$$\mathbf{x} \mapsto \mathbf{R}_\theta \mathbf{x} \quad \leftrightarrow \quad (r, \alpha) \mapsto (r, \alpha + \theta)$$

Proof:  $\mathbf{R}_\theta \mathbf{x} = \mathbf{R}_\theta \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$

$$= \begin{pmatrix} r(\cos \theta \cos \alpha - \sin \alpha \sin \theta) \\ r(\cos \theta \sin \alpha + \cos \alpha \sin \theta) \end{pmatrix}$$
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- Then, functions are rotated simply by a shift in the angular axis

$$\mathcal{L}_\theta^{SO(2)} f(\mathbf{x}) = f(\mathbf{R}_\theta^{-1} \mathbf{x}) \quad \leftrightarrow \quad \mathcal{L}_\theta^{SO(2)} \bar{f}(r, \alpha) = \bar{f}(r, \alpha - \theta)$$

Proof:  $\mathbf{R}_\theta \mathbf{x} = \mathbf{R}_\theta \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$

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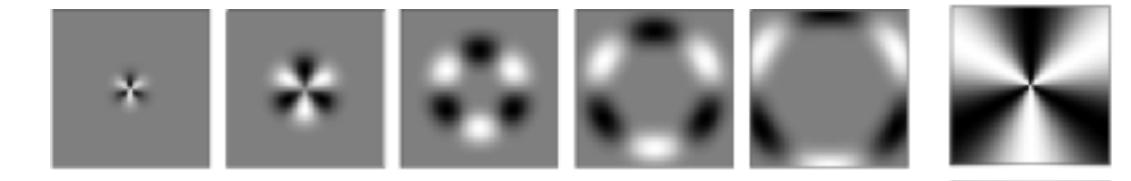
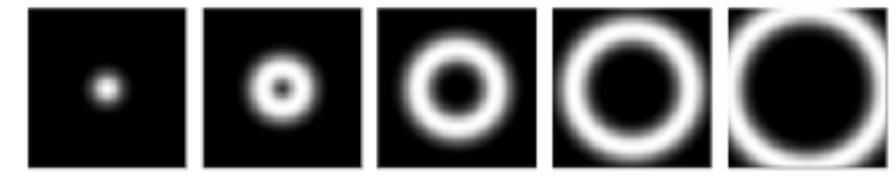
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- Now, let's use this to parametrize polar-separable conv kernels and focus on the angular part

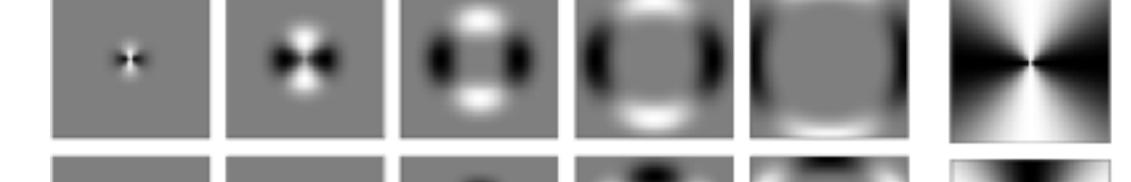
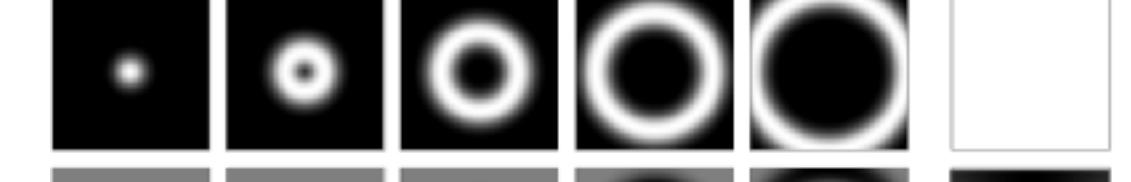
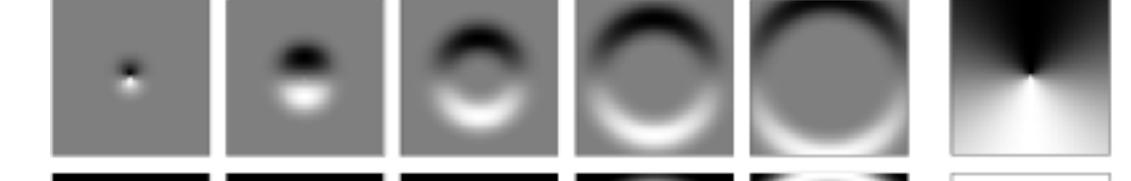
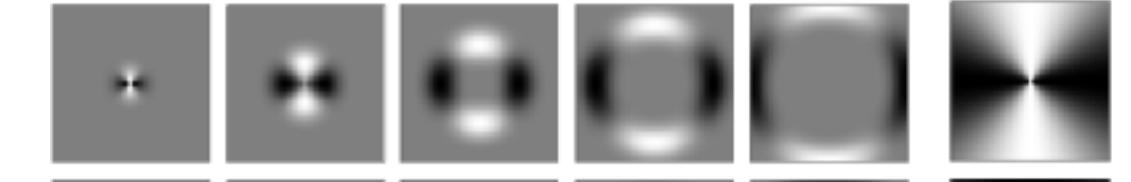
$$k(\mathbf{x} \mid \mathbf{w}) = k^\rightarrow(r \mid \mathbf{w}) k^\circlearrowleft(\alpha \mid \mathbf{w})$$

A function on  $S^1$  !!!

$$k_m^\rightarrow(r)$$



$$k_l^\circlearrowleft(\alpha)$$



# Two dimensional rotation-steerable functions

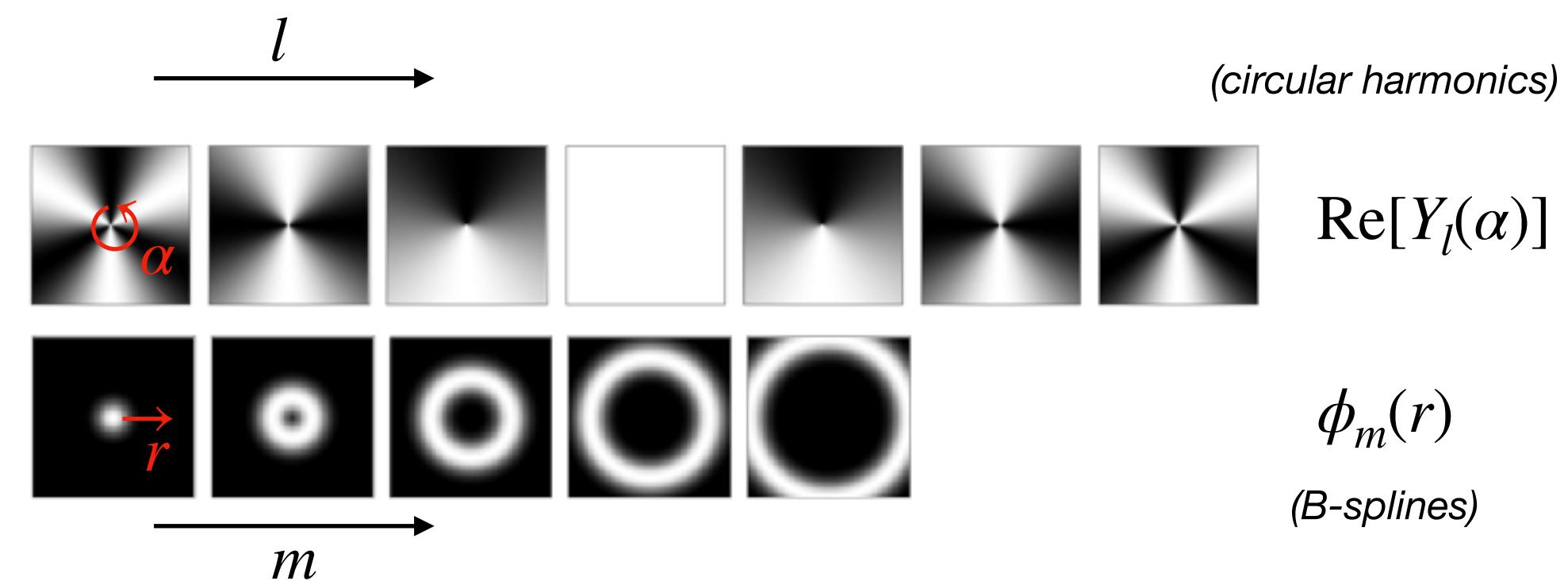
- Consider polar-separable convolution kernel:

$$k(\mathbf{x} \mid \mathbf{w}) = k^\rightarrow(r \mid \mathbf{w}) k^\circlearrowleft(\alpha \mid \mathbf{w}),$$

- with  $k^\circlearrowleft$  in an  $SO(2)$  steerable basis, and  $k^\rightarrow$  in some radial basis:

$$k^\circlearrowleft(\alpha \mid \mathbf{w}) = \sum_l \bar{w}_l Y_l(\alpha), \quad \text{e.g., with} \quad Y_l(\alpha) = e^{i l \alpha},$$

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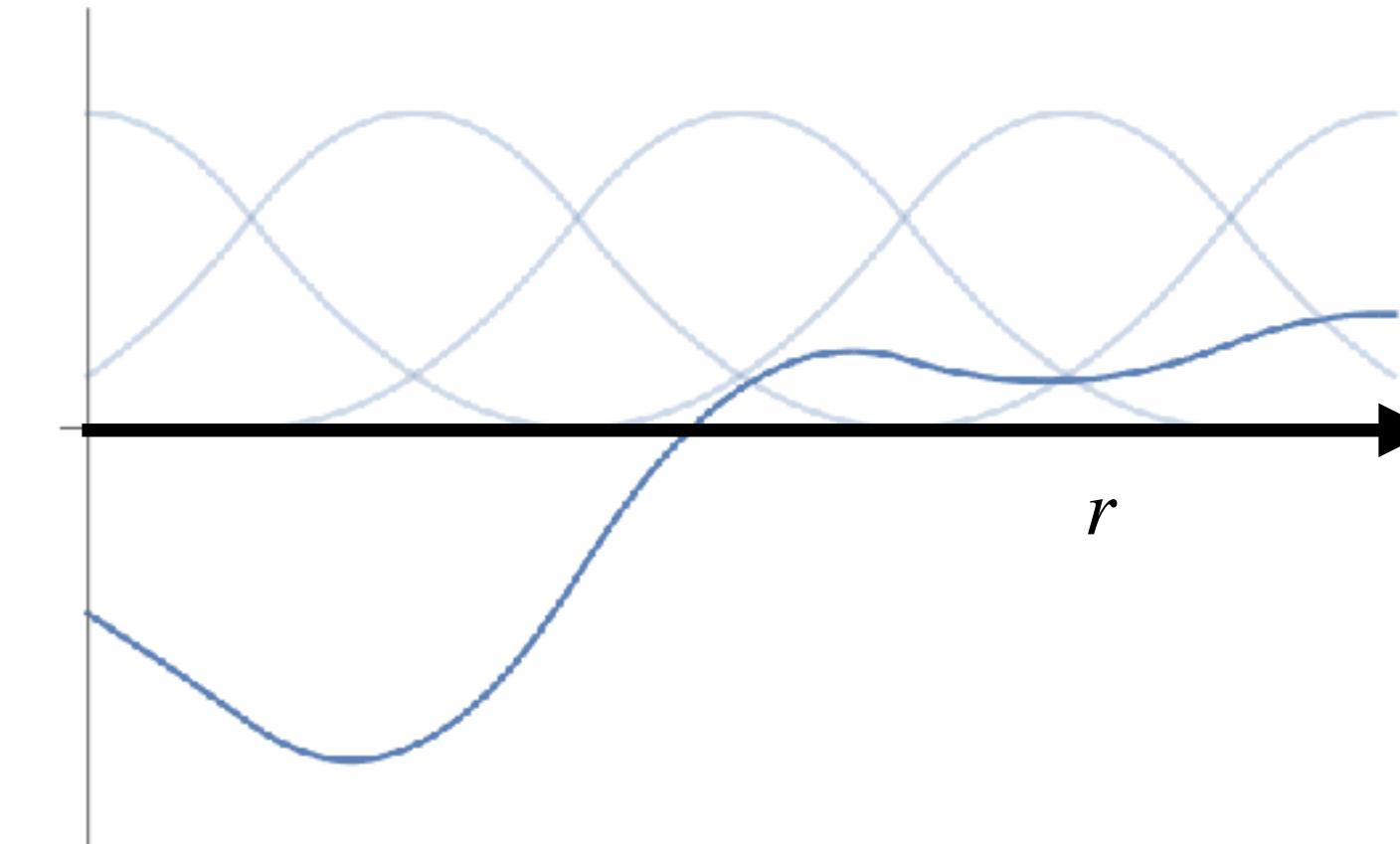
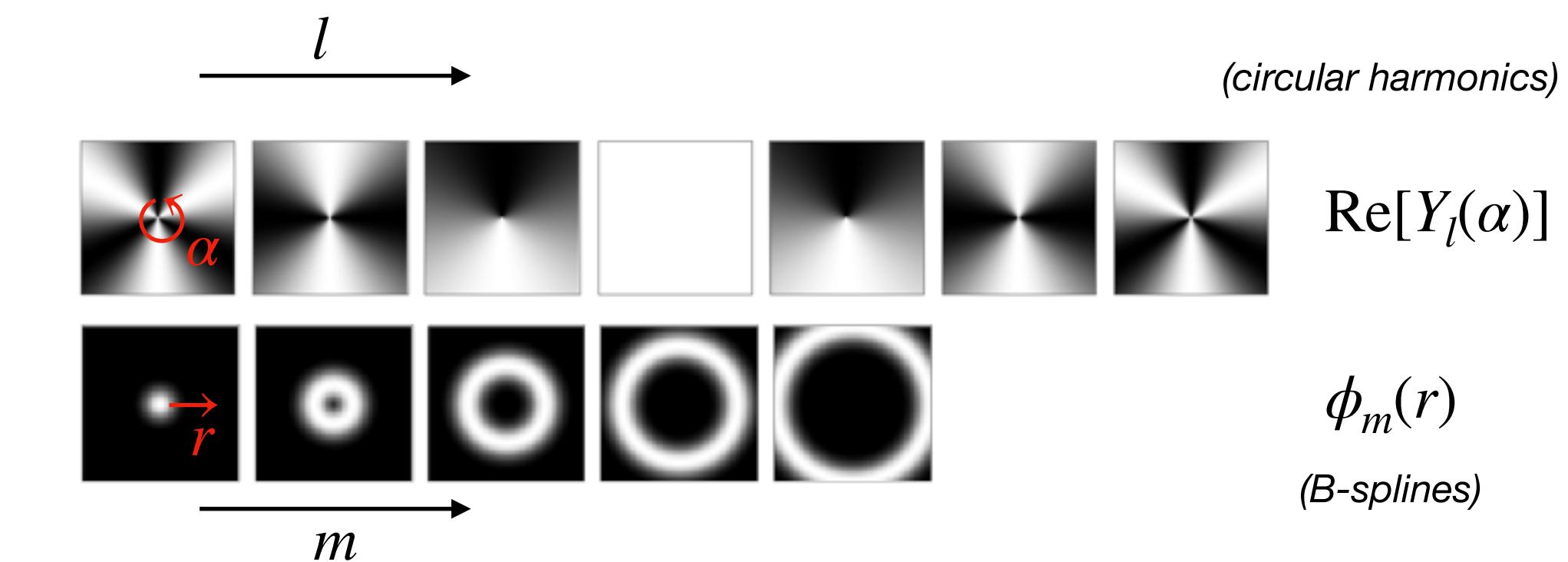
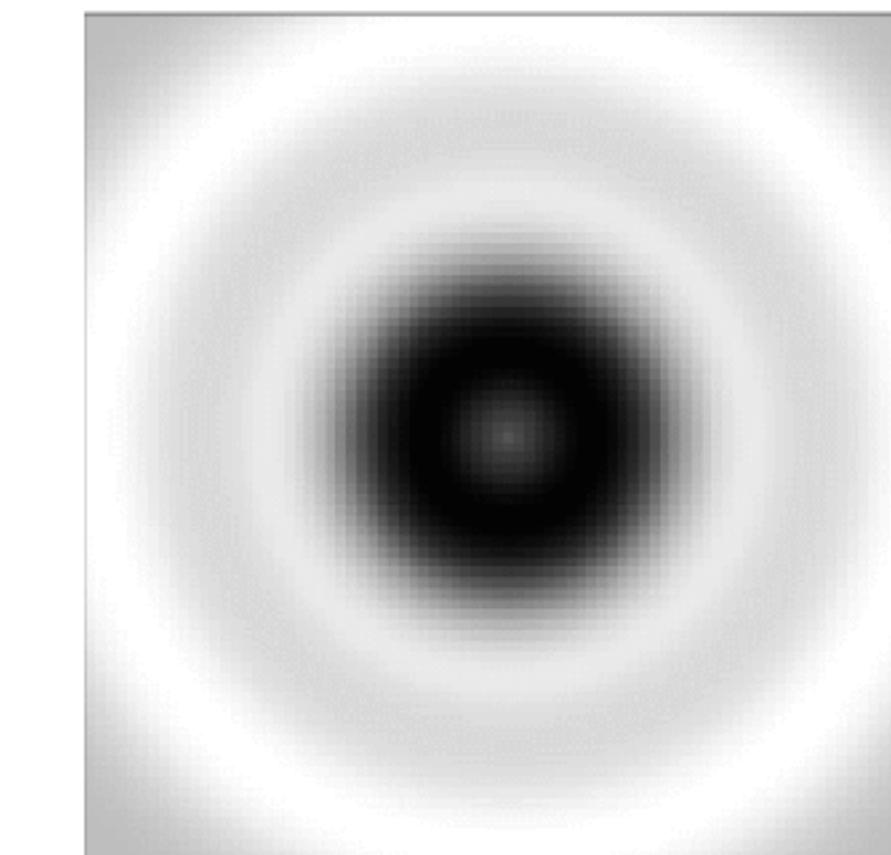
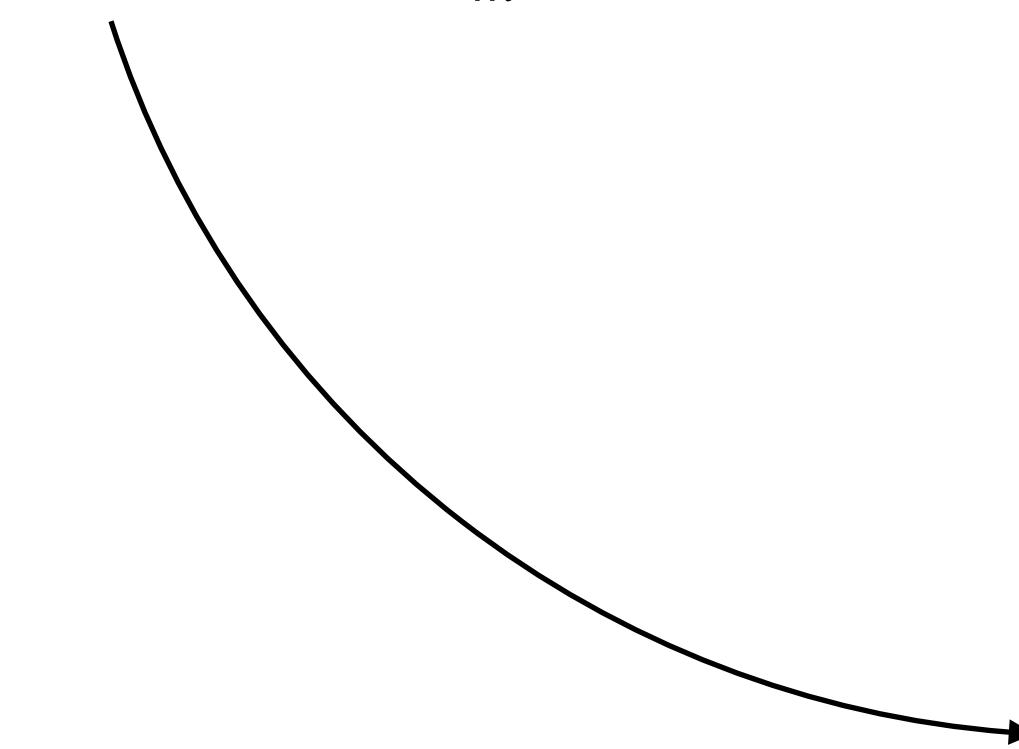
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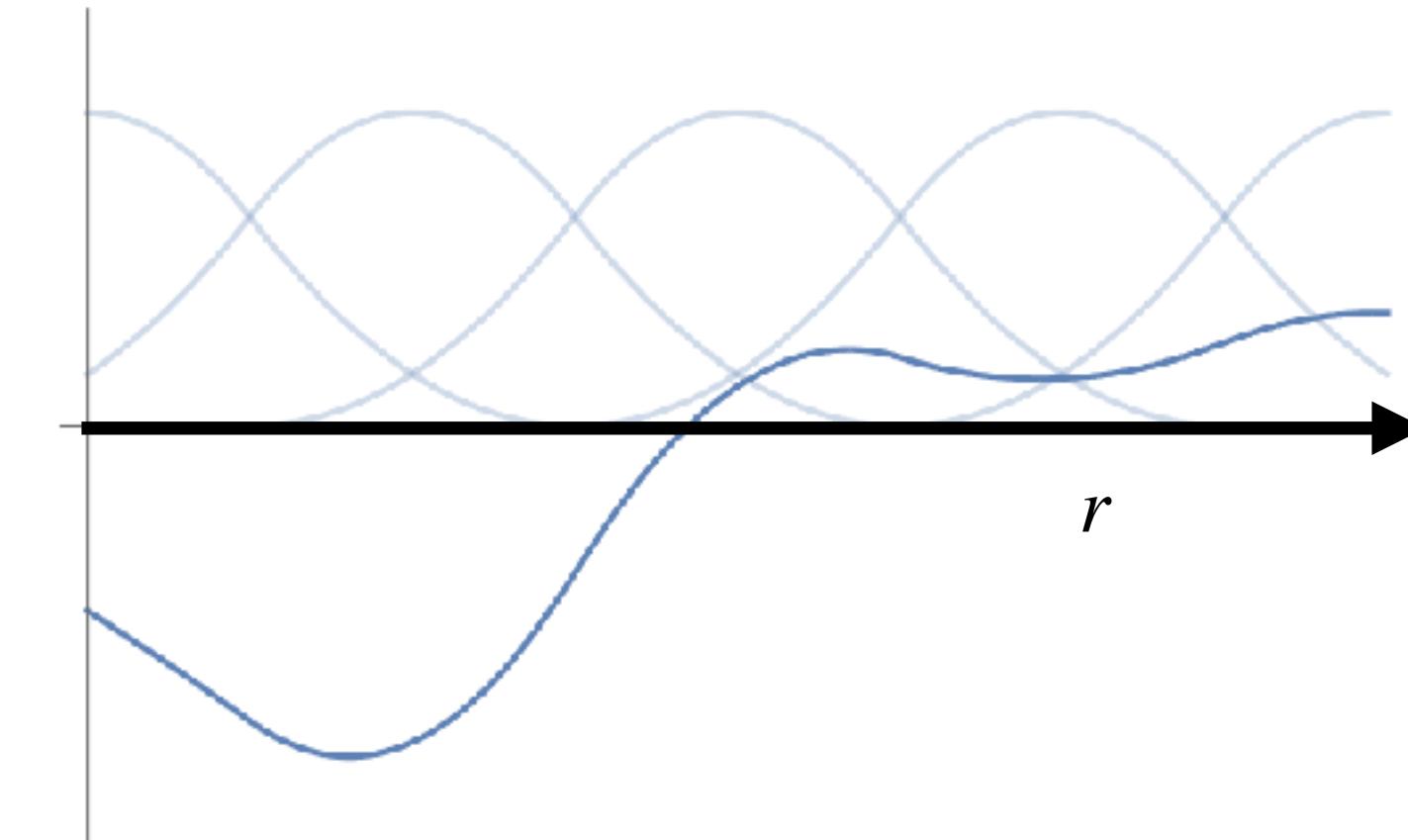
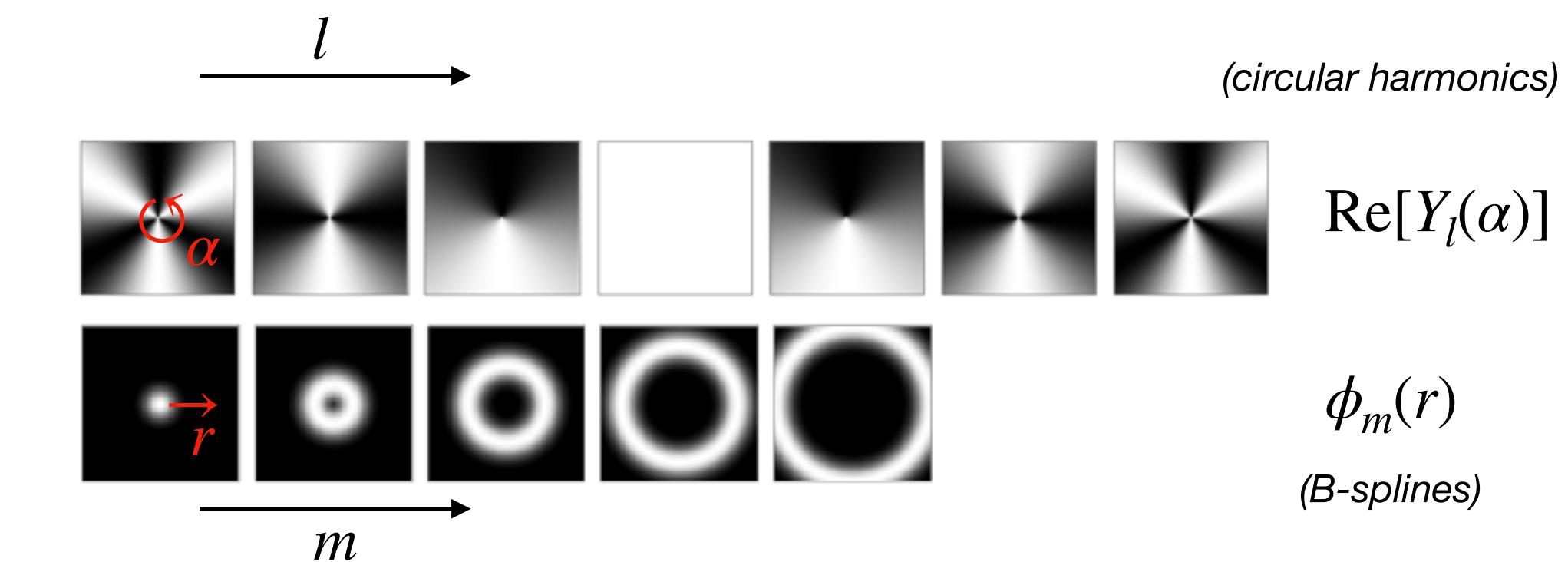
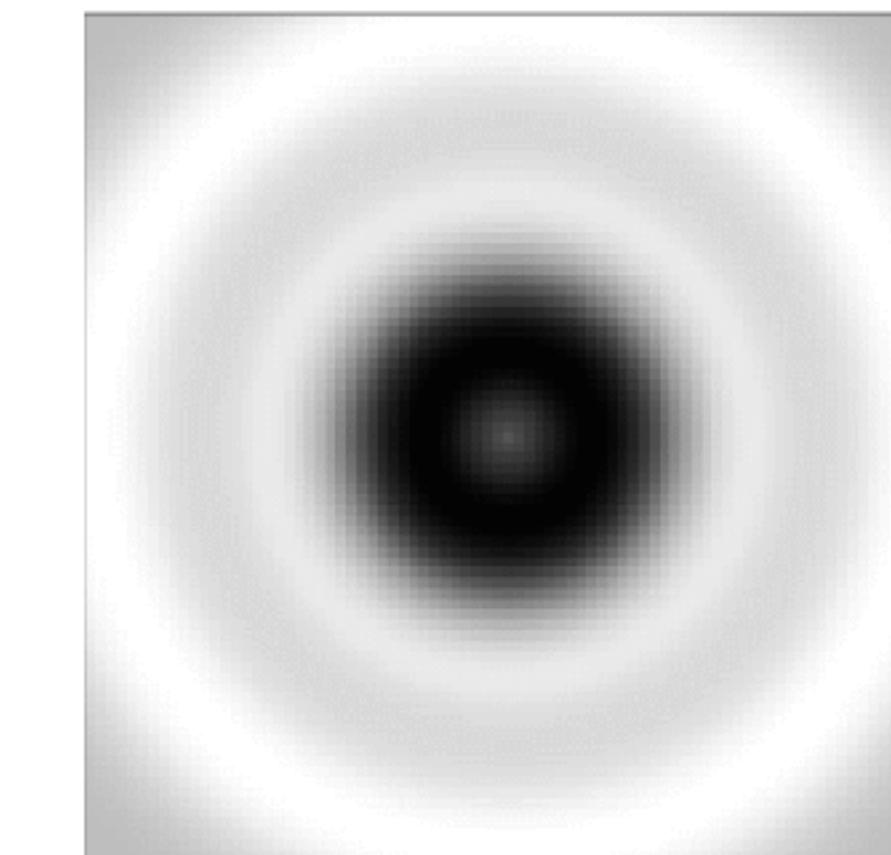
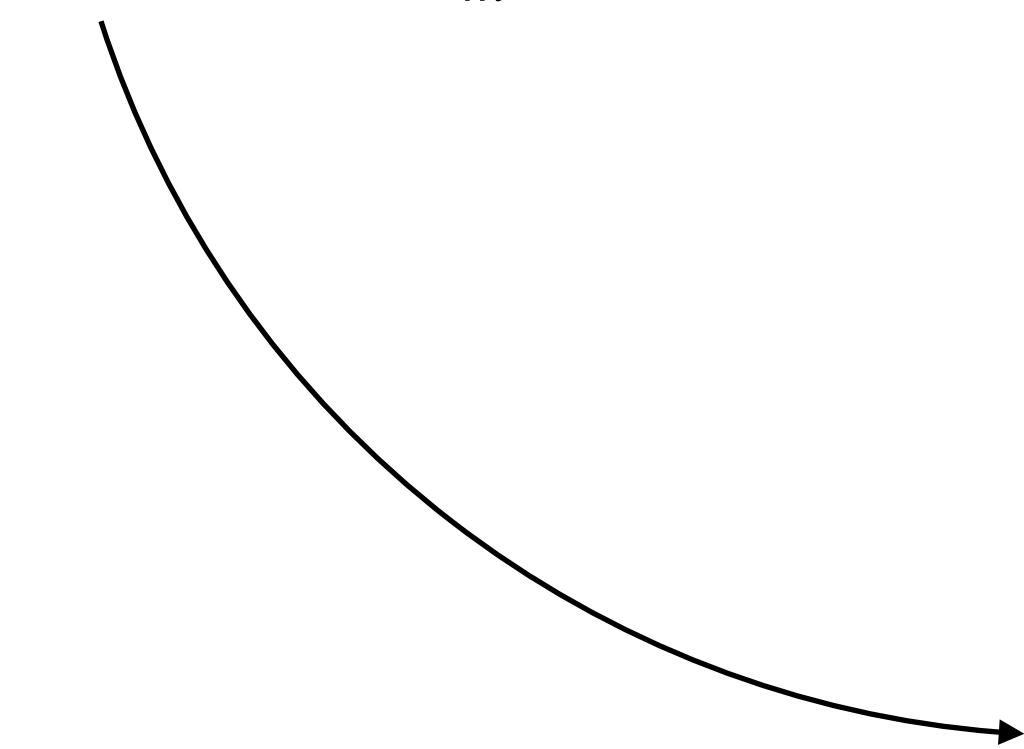
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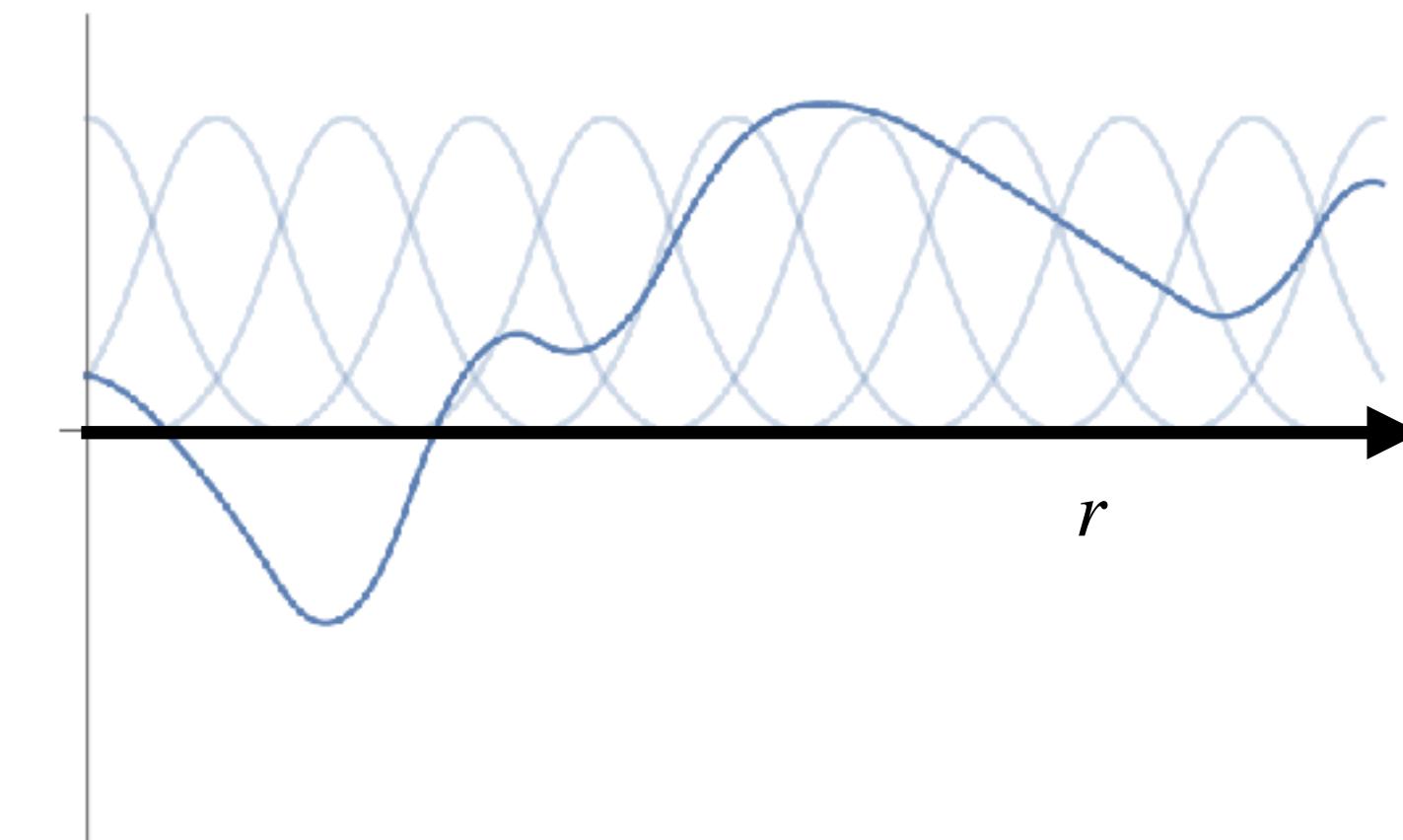
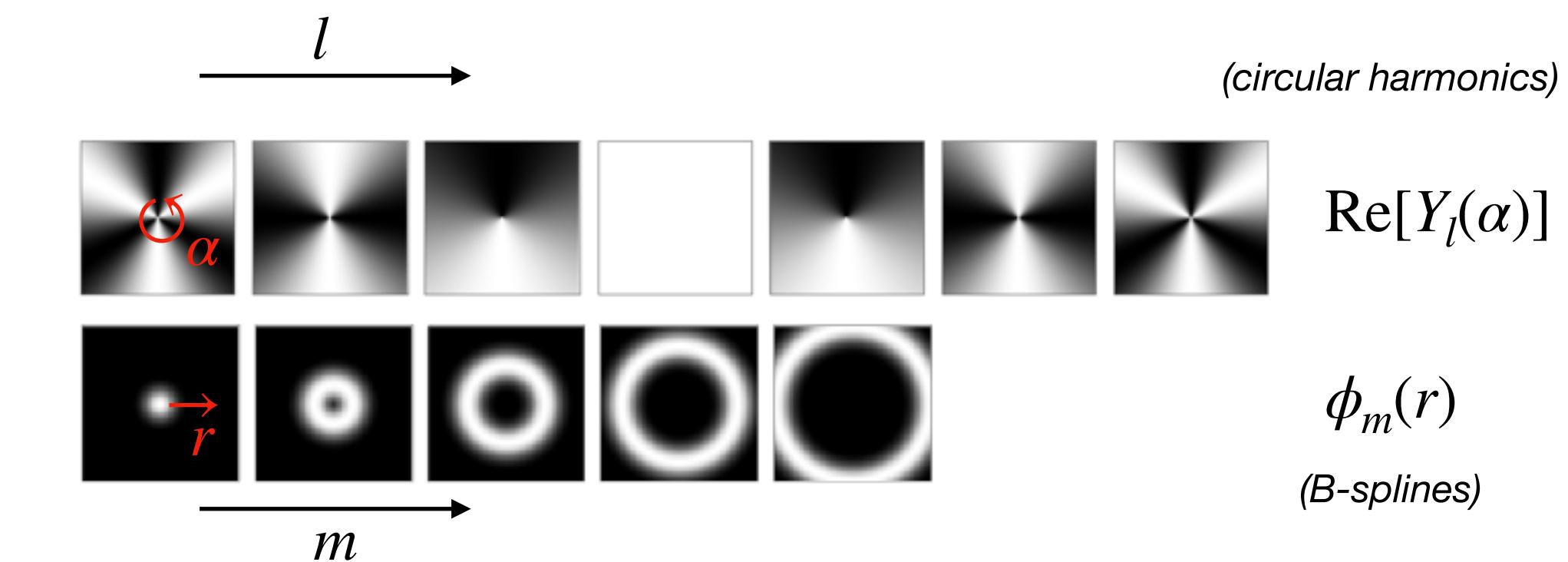
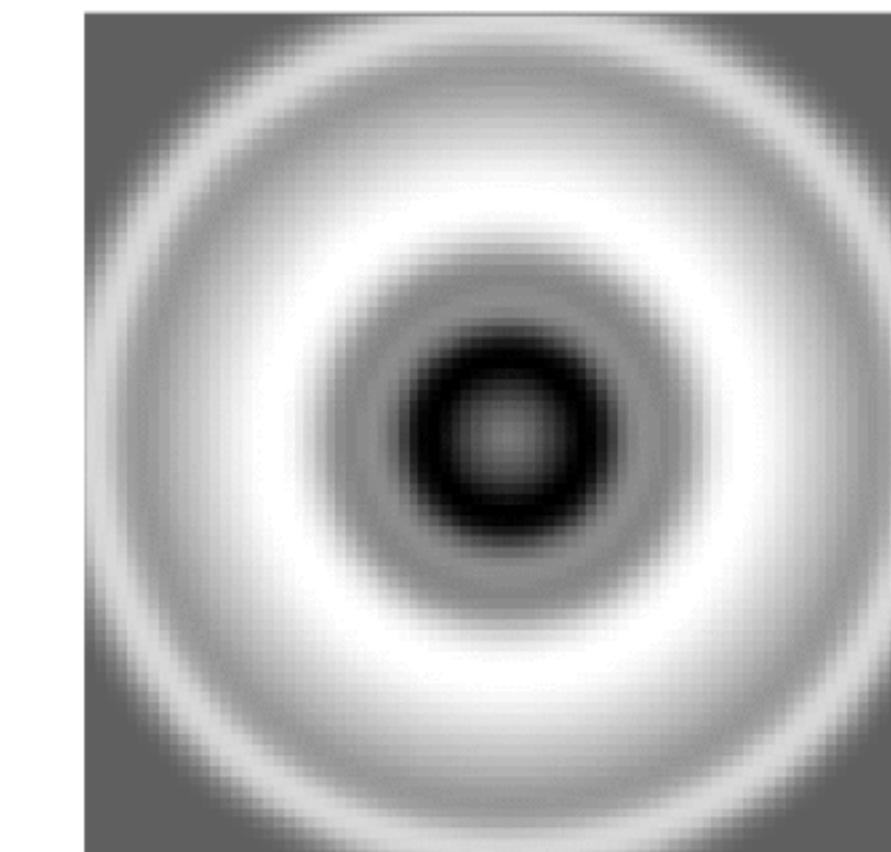
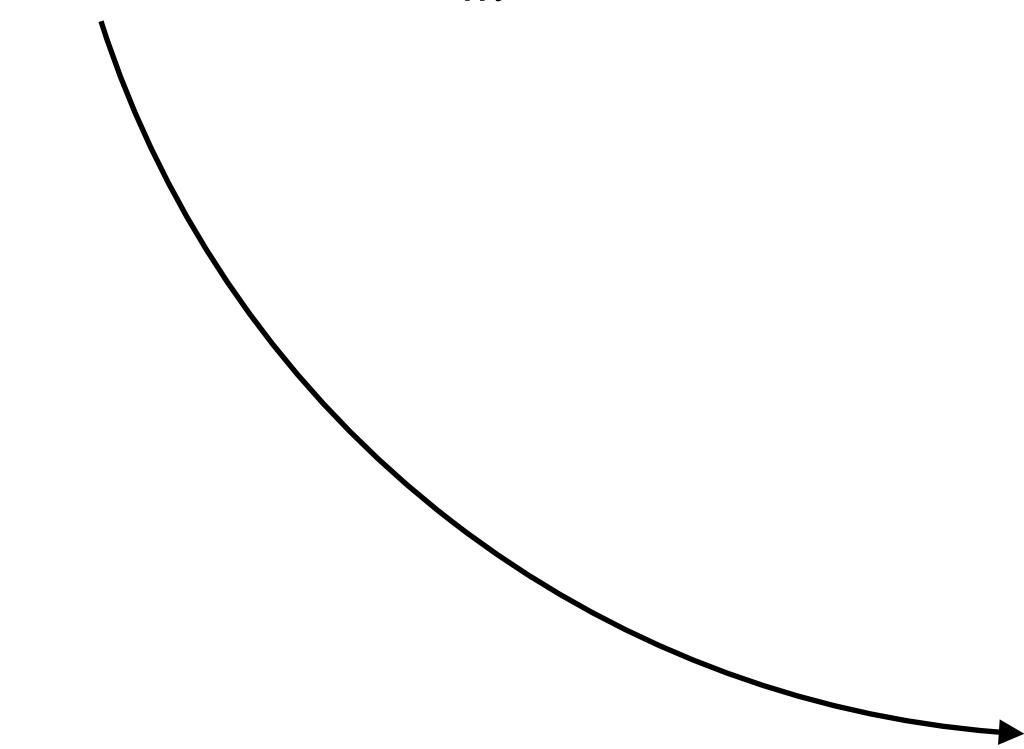
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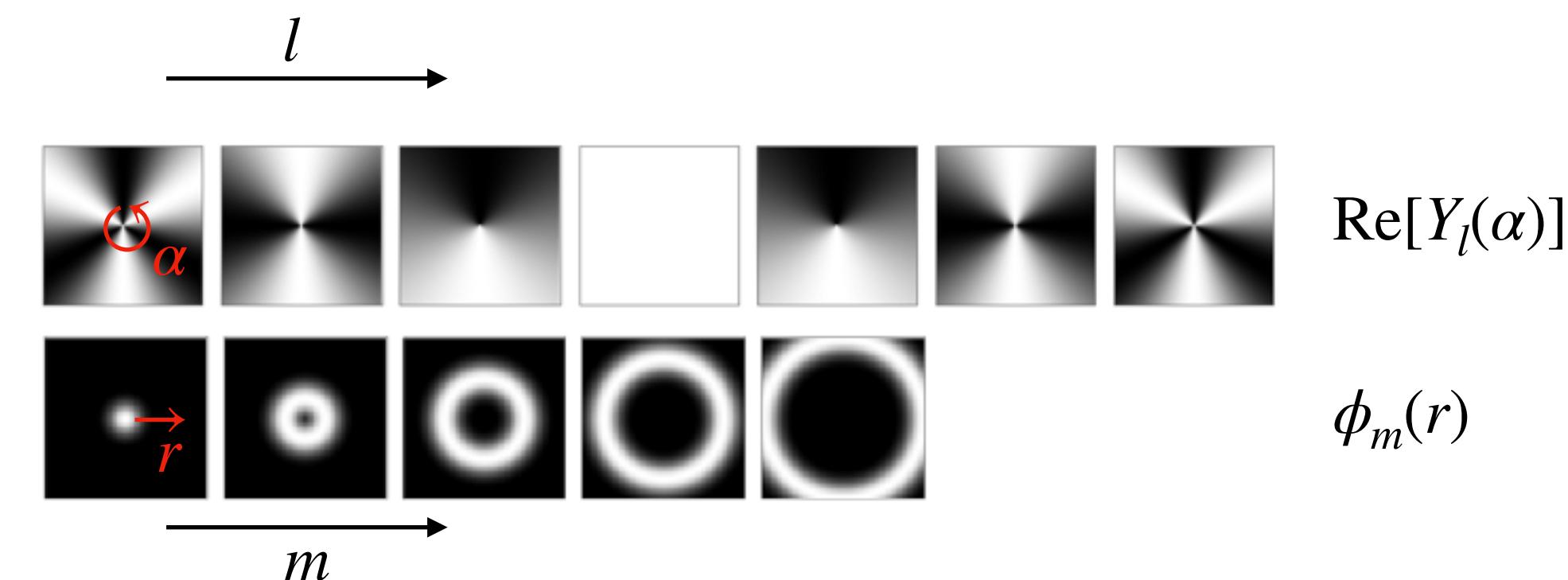
- Then we may as well write it as

$$\begin{aligned} k(\mathbf{x} | \mathbf{w}) &= \sum_l \sum_m w_m \bar{w}_l \phi_m(r) Y_l(\alpha) \\ &= \sum_l \sum_m \bar{w}_{ml} \phi_m(r) Y_l(\alpha) \quad (\text{"absorb" weights}) \\ &= \sum_l \hat{w}_l(r) Y_l(\alpha) \end{aligned}$$

with radius dependent weights  $\hat{w}_l(r) = \sum_m w_{ml} \phi_m(r)$

- Then such kernel is clearly rotation steerable!

$$k(\mathbf{R}_\theta^{-1} \mathbf{x} | \hat{\mathbf{w}}(r)) = k(\mathbf{x} | \rho(\theta) \hat{\mathbf{w}}(r))$$



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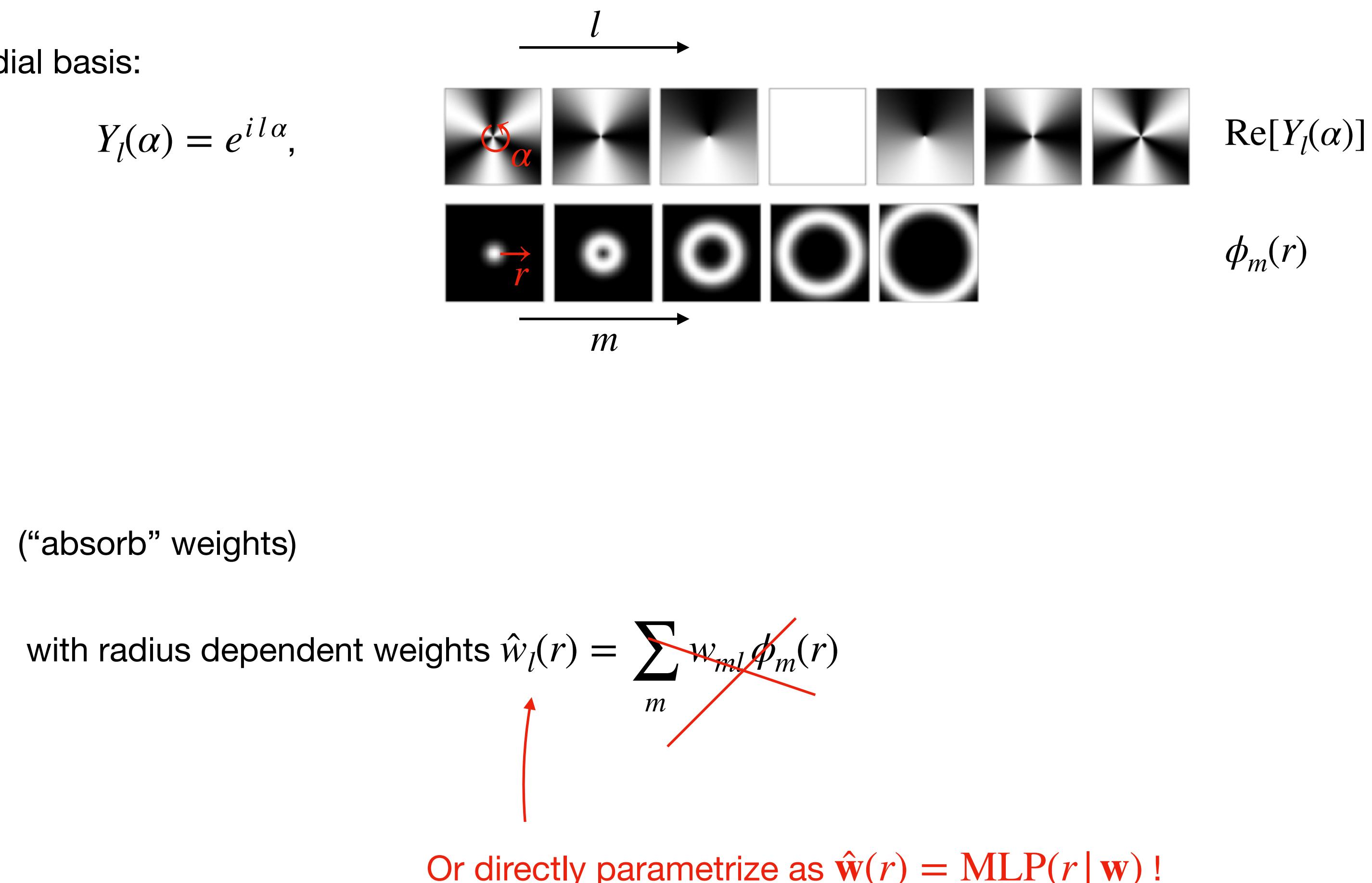
$$k^\rightarrow(r \mid \mathbf{w}) = \sum_m w_m \phi_m(r)$$

- Then we may as well write it as

$$\begin{aligned} k(\mathbf{x} \mid \mathbf{w}) &= \sum_l \sum_m w_m \bar{w}_l \phi_m(r) Y_l(\alpha) \\ &= \sum_l \sum_m \bar{w}_{ml} \phi_m(r) Y_l(\alpha) \quad (\text{"absorb" weights}) \\ &= \sum_l \bar{w}_l(r) Y_l(\alpha) \end{aligned}$$

- Then such kernel is clearly rotation steerable!

$$k(\mathbf{R}_\theta^{-1} \mathbf{x} \mid \hat{\mathbf{w}}(r)) = k(\mathbf{x} \mid \rho(\theta) \hat{\mathbf{w}}(r))$$



# Complex (irreducible) representations

$$\begin{array}{ccc}
 Y(\mathbf{R}_\theta^{-1} \mathbf{x}) & = & \rho(\mathbf{R}_\theta^{-1}) \\
 \begin{array}{cc}
 \text{Re} & \text{Im} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) \\
 \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & = & \left( \begin{array}{cccccc} e^{3i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{1i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta} \end{array} \right) \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & & \left( \begin{array}{cc}
 \text{Re} & \text{Im} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) \\
 \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & & \left( \begin{array}{cc} \text{Re} & \text{Im} \\ \text{Im} & \text{Re} \end{array} \right)
 \end{array} \right)
 \end{array}$$

# Complex (irreducible) representations

$$\begin{array}{ccc}
 Y(\mathbf{R}_\theta^{-1} \mathbf{x}) & = & \rho(\mathbf{R}_\theta^{-1}) \\
 \begin{array}{cc}
 \text{Re} & \text{Im} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) \\
 \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & = & \left( \begin{array}{cccccc} e^{3i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{1i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta} \end{array} \right) \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & & \left( \begin{array}{cc}
 \text{Re} & \text{Im} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) \\
 \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \\ \text{Re} \\ \text{Im} \end{array} \right) & & \left( \begin{array}{cc} \text{Re} & \text{Im} \\ \text{Im} & \text{Re} \end{array} \right)
 \end{array} \right)
 \end{array}$$

# Complex (irreducible) representations

$$\begin{array}{ccc}
 Y(\mathbf{R}_\theta^{-1} \mathbf{x}) & = & \rho(\mathbf{R}_\theta^{-1}) \\
 \begin{array}{cc}
 \text{Re} & \text{Im} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \vdots \\ \text{Re} \\ \text{Im} \end{array} \right) & \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \vdots \\ \text{Re} \\ \text{Im} \end{array} \right) \\
 \longrightarrow & \longrightarrow
 \end{array} & = & \begin{pmatrix} e^{3i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{1i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix} \\
 \begin{array}{cc}
 \text{Re} & \text{Im} \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \vdots \\ \text{Re} \\ \text{Im} \end{array} \right) & \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \vdots \\ \text{Re} \\ \text{Im} \end{array} \right) \\
 \longrightarrow & \longrightarrow
 \end{array} & = & \begin{array}{cc}
 Y(\mathbf{x}) \\
 \left( \begin{array}{c} \text{Re} \\ \text{Im} \\ \vdots \\ \text{Re} \\ \text{Im} \end{array} \right) \\
 \longrightarrow
 \end{array}
 \end{array}$$

cos( $l\alpha$ )

sin( $l\alpha$ )

# Real (irreducible) representations

$$\begin{aligned}
 Y(\mathbf{R}_\theta^{-1} \mathbf{x}) &= \rho(\mathbf{R}_\theta^{-1}) Y(\mathbf{x}) \\
 \left( \begin{array}{c} \text{Image 1} \\ \text{Image 2} \\ \text{Image 3} \\ \text{Image 4} \\ \text{Image 5} \\ \text{Image 6} \\ \text{Image 7} \\ \text{Image 8} \end{array} \right) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin \theta & \cos \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 2\theta & \sin 2\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta & 0 \\ 0 & 0 & 0 & 0 & -\sin 3\theta & 0 & \cos 3\theta & 0 \end{pmatrix} \left( \begin{array}{c} \text{Image 1} \\ \text{Image 2} \\ \text{Image 3} \\ \text{Image 4} \\ \text{Image 5} \\ \text{Image 6} \\ \text{Image 7} \\ \text{Image 8} \end{array} \right)
 \end{aligned}$$

# Real (irreducible) representations

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 \end{aligned}$$

# Real (irreducible) representations

The real basis functions  $Y_l(\mathbf{x}) = \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix}$  are steerable using  $\rho_l(\mathbf{R}_\theta) = \begin{pmatrix} \cos l\theta & -\sin l\theta \\ \sin l\theta & \cos l\theta \end{pmatrix}$

## Proof:

$$\begin{aligned}
Y_l(\mathbf{R}_\theta^{-1} \mathbf{x}) &= \begin{pmatrix} \cos(l(\alpha - \theta)) \\ \sin(l(\alpha - \theta)) \end{pmatrix} \\
&= \begin{pmatrix} \cos(l\alpha + -l\theta) \\ \sin(l\alpha + -l\theta) \end{pmatrix} = \begin{pmatrix} \cos(l\alpha)\cos(-l\theta) - \sin(l\alpha)\sin(-l\theta) \\ \sin(l\alpha)\cos(-l\theta) + \cos(l\alpha)\sin(-l\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos -l\theta & -\sin -l\theta \\ \sin -l\theta & \cos -l\theta \end{pmatrix} \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix} \\
&= \rho_l(\mathbf{R}_\theta^{-1}) Y_l(\mathbf{x})
\end{aligned}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\
0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta
\end{pmatrix}$$

# Real (irreducible) representations

The real basis functions  $Y_l(\mathbf{x}) = \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix}$  are steerable using  $\rho_l(\mathbf{R}_\theta) = \begin{pmatrix} \cos l\theta & -\sin l\theta \\ \sin l\theta & \cos l\theta \end{pmatrix}$

Proof:

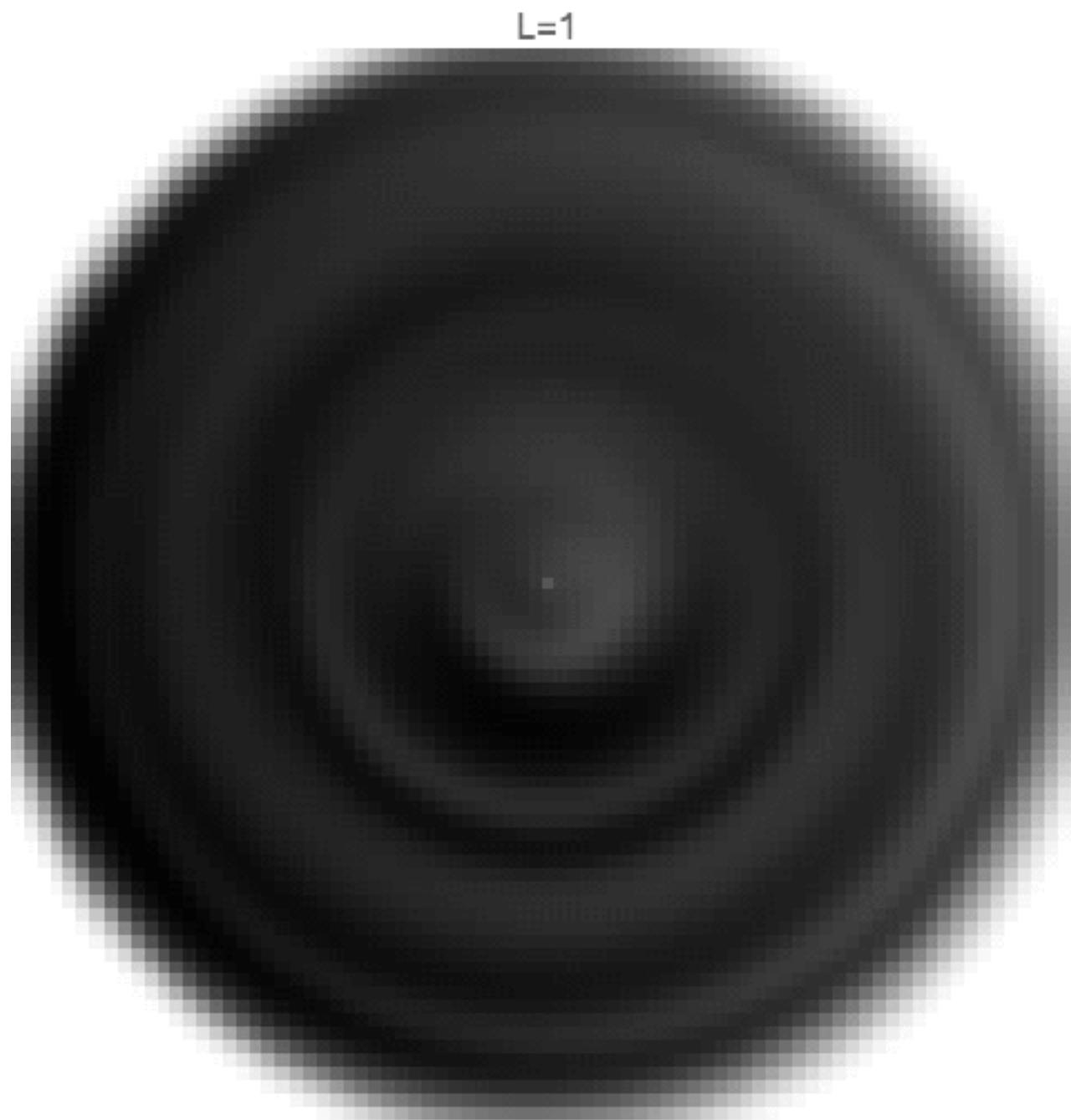
$$\begin{aligned} Y_l(\mathbf{R}_\theta^{-1}\mathbf{x}) &= \begin{pmatrix} \cos(l(\alpha - \theta)) \\ \sin(l(\alpha - \theta)) \end{pmatrix} \\ &= \begin{pmatrix} \cos(l\alpha + -l\theta) \\ \sin(l\alpha + -l\theta) \end{pmatrix} = \begin{pmatrix} \cos(l\alpha)\cos(-l\theta) - \sin(l\alpha)\sin(-l\theta) \\ \sin(l\alpha)\cos(-l\theta) + \cos(l\alpha)\sin(-l\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos -l\theta & -\sin -l\theta \\ \sin -l\theta & \cos -l\theta \end{pmatrix} \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix} \\ &= \rho_l(\mathbf{R}_\theta^{-1}) Y_l(\mathbf{x}) \end{aligned}$$

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta & \cos 3\theta & \sin 3\theta \\ \sin 2\theta & \cos 2\theta & -\sin 3\theta & \cos 3\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

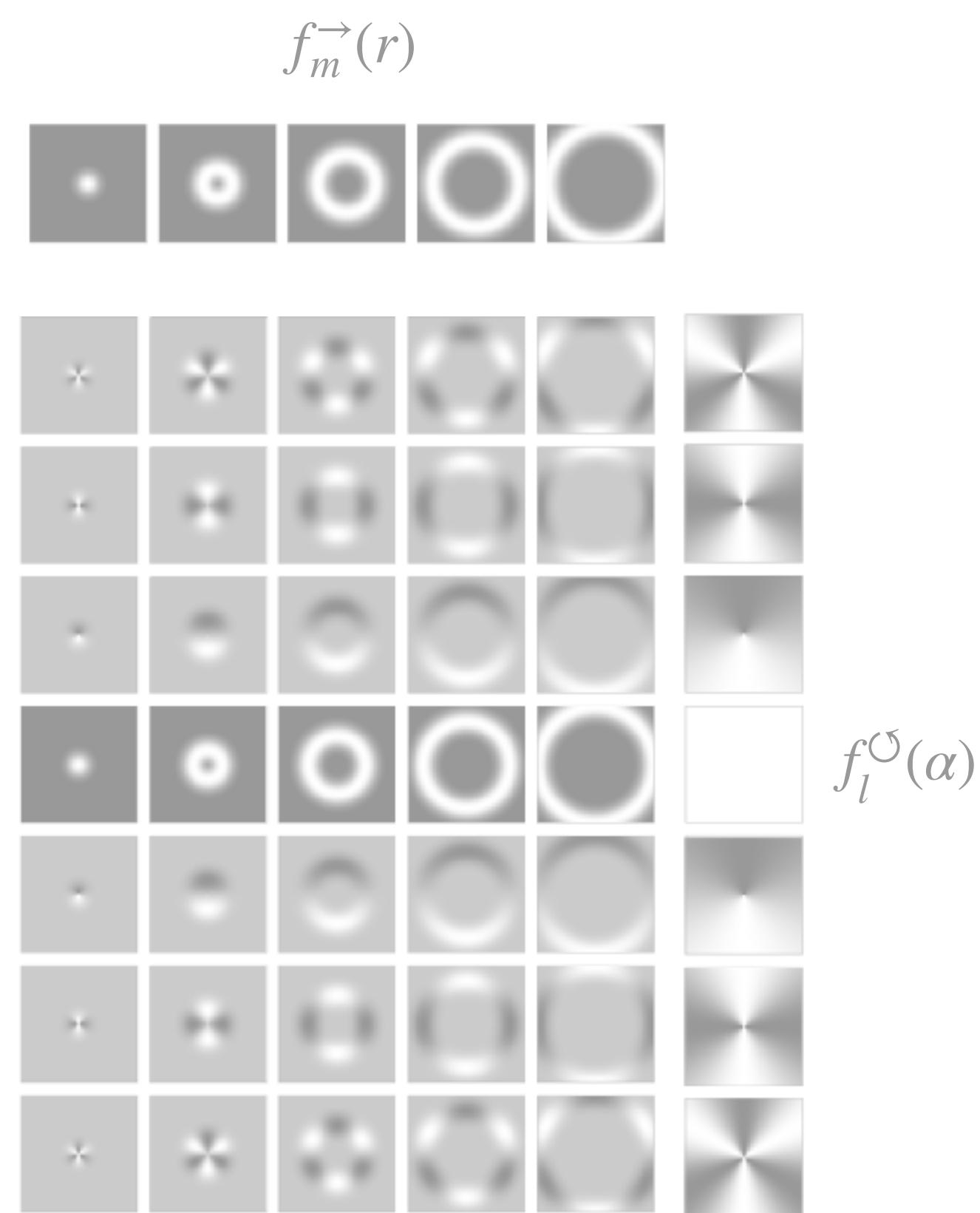
# Representing interesting convolution kernels in a steerable basis!

## Exercise:

1. Tune the weights  $\hat{\mathbf{w}}$  until you get something interesting.
2. Add more detail by increasing maximum frequency!



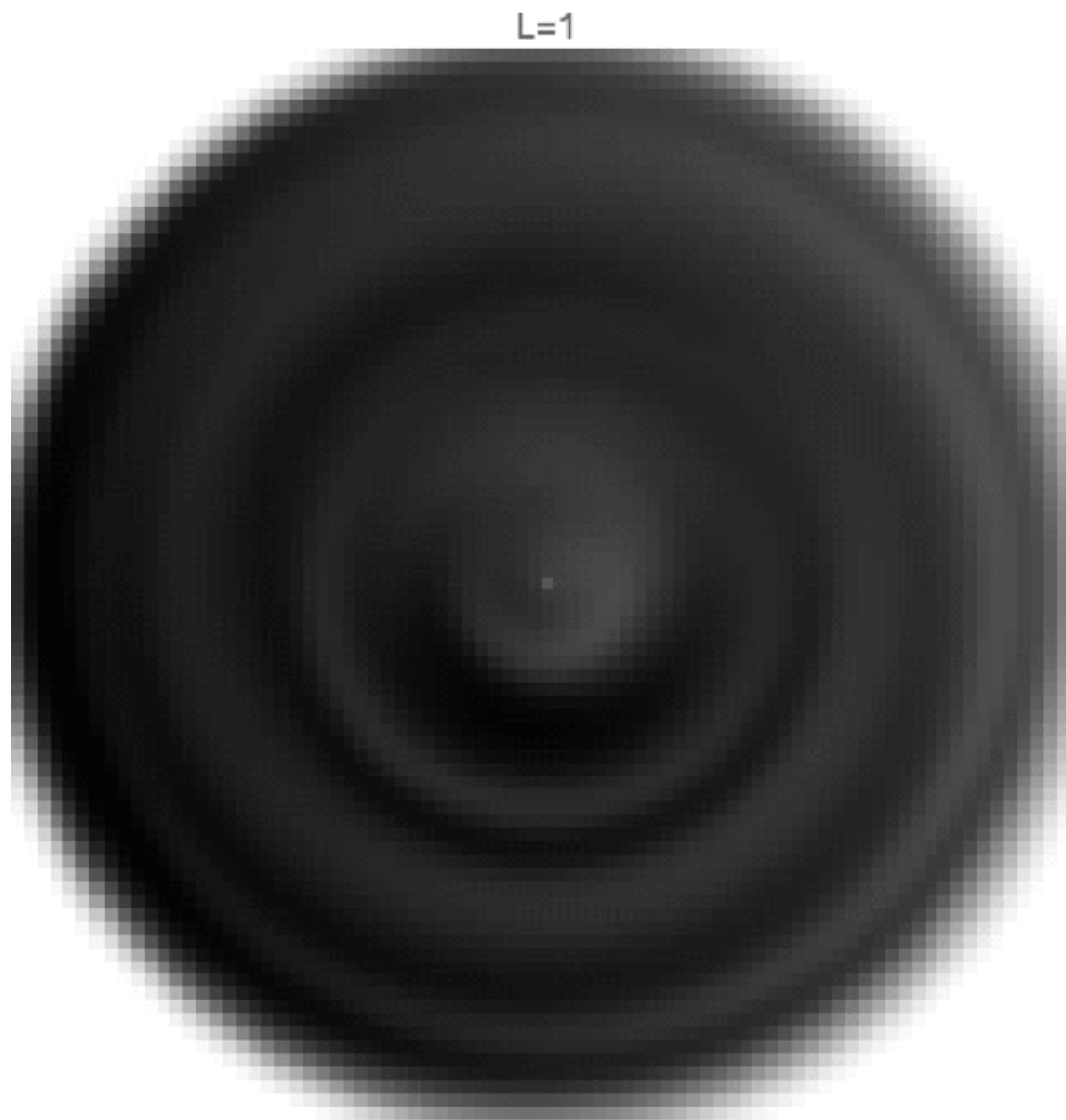
$$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$$



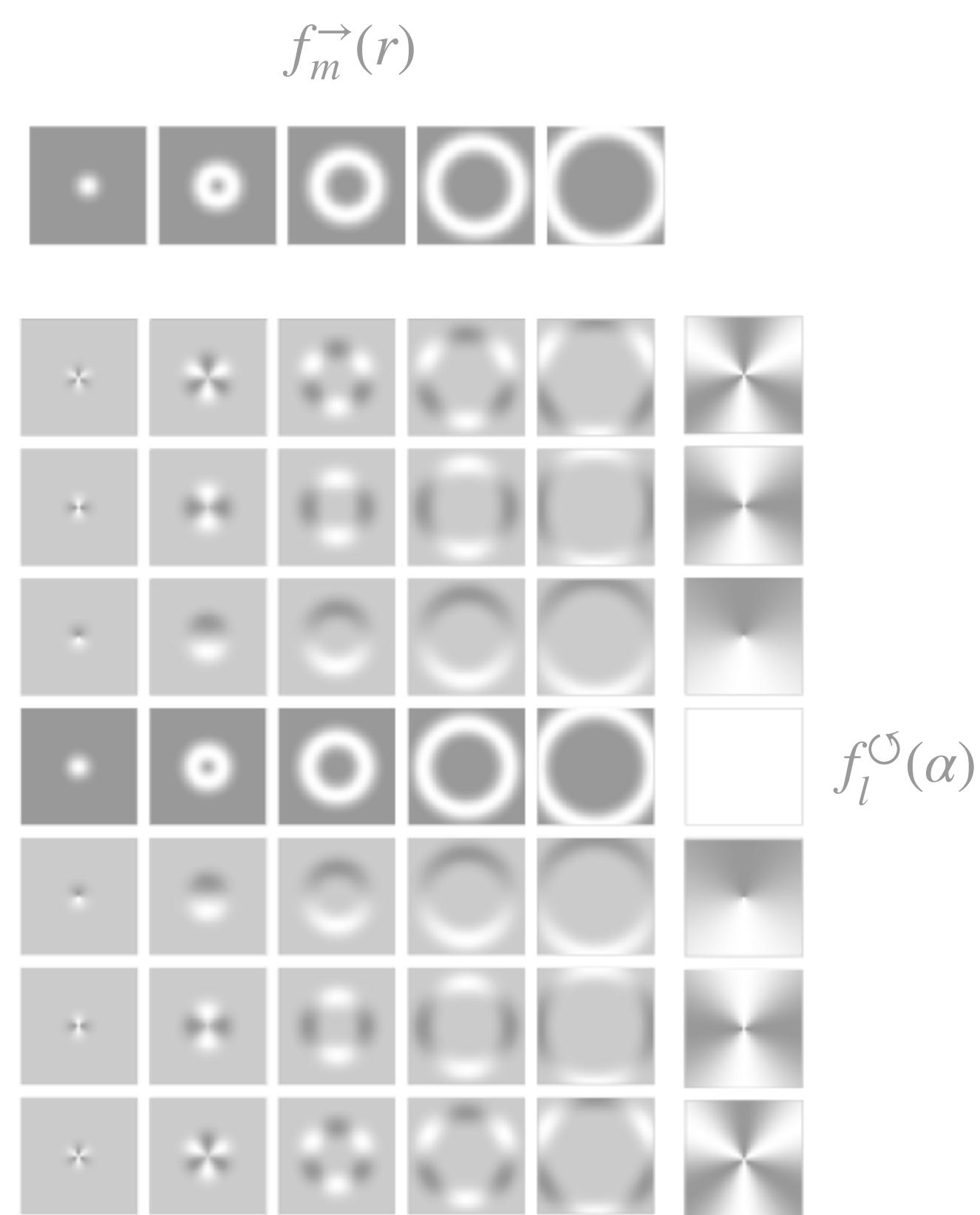
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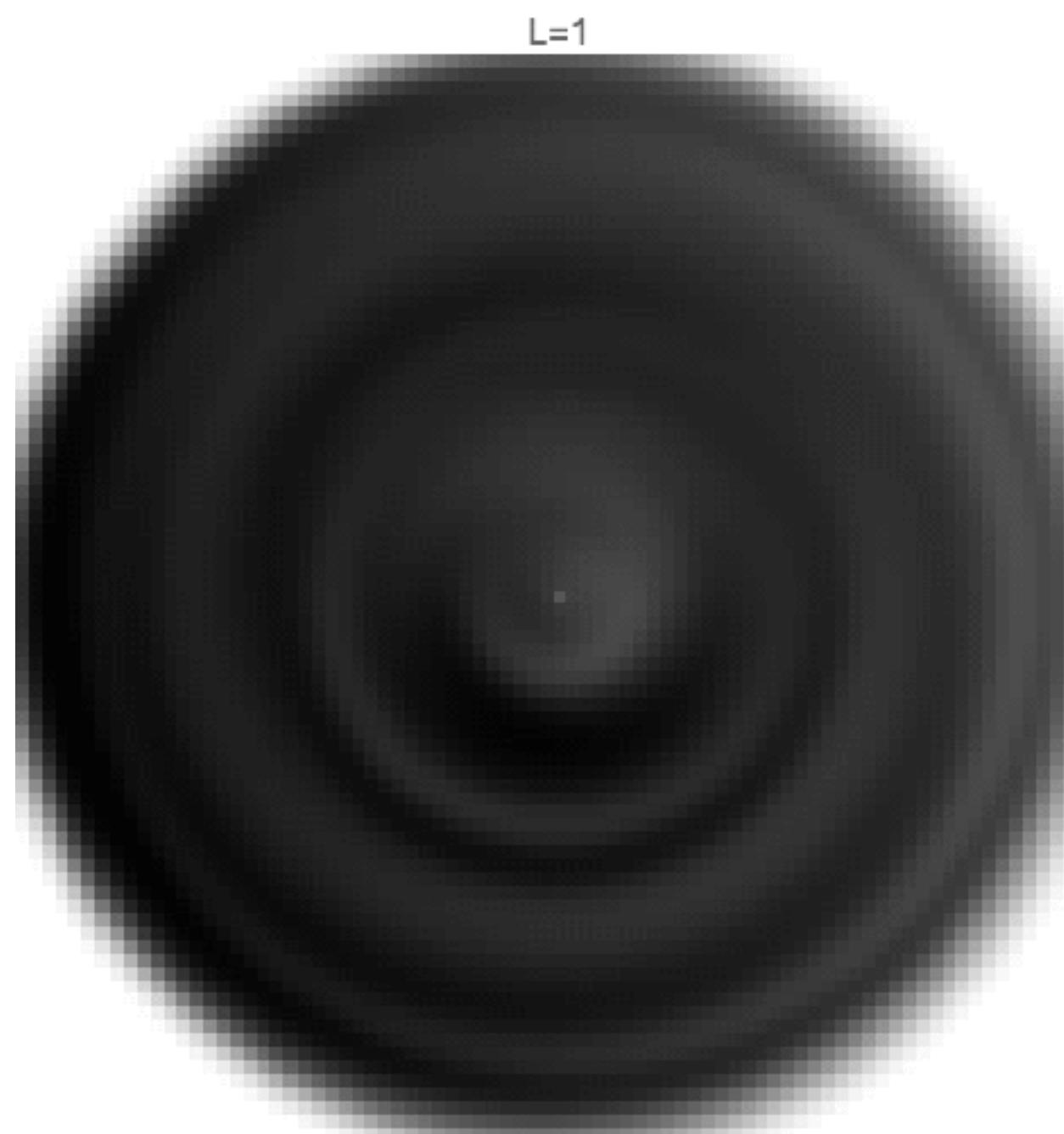
$$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$$



# Representing interesting convolution kernels in a steerable basis!

## Exercise:

1. Tune the weights  $\hat{\mathbf{w}}$  until you get something interesting.
2. Add more detail by increasing maximum frequency!
3. Go crazy and **steer** it by transforming the weights!



$$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$$



$$k(\mathbf{x} \mid \rho(\theta)\hat{\mathbf{w}}(r))$$

